

# Renormalization-group analysis of the validity of staggered-fermion QCD with the fourth-root recipe

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## Abstract

I develop a renormalization-group blocking framework for lattice QCD with staggered fermions. Under plausible, and testable, assumptions, I then argue that the fourth-root recipe used in numerical simulations is valid in the continuum limit. The taste-symmetry violating terms, which give rise to non-local effects in the fourth-root theory when the lattice spacing is non-zero, vanish in the continuum limit. A key role is played by reweighted theories that are local and renormalizable on the one hand, and that approximate the fourth-root theory better and better as the continuum limit is approached on the other hand.

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## I. INTRODUCTION

Lattice QCD simulations with staggered fermions [1, 2, 3, 4, 5, 6, 7, 8] have been producing remarkably accurate predictions of various hadronic observables [9]. The staggered-fermion field has only one component per color per lattice site, making the numerical computations relatively cheap, as well as a non-anomalous  $U(1)$  chiral symmetry in the massless limit, which is important for the phenomenology of the light-quark sector.

All staggered-fermion simulations with three flavors of light quarks make use of the fourth-root recipe [10]. The up, down, and strange quarks are each represented by a staggered field with a different bare mass. But normally a single staggered field generates four quark species, or “tastes,” in the continuum limit. The four tastes do have equal renormalized masses thanks to the lattice staggered-fermion symmetries [8].<sup>1</sup> In order to remove the excessive degrees of freedom one takes the fourth root of the staggered-fermion determinant.<sup>2</sup> The fourth-root recipe defines a renormalizable theory which, to all orders in perturbation theory, reproduces a local, unitary theory with the correct number of light quarks in the limit of a vanishing lattice spacing. (It is assumed [12] that the staggered theory without the fourth root behaves as expected in perturbation theory, which is very plausible [11, 13].) Non-perturbatively, the validity of the fourth-root recipe is a non-trivial issue which has been the subject of much debate [14]. For a recent review, see Ref. [11].

In a formal expansion in the lattice spacing, the massless staggered action splits into marginal terms that have a  $U(4)$  taste symmetry and irrelevant terms that break the symmetry explicitly. Because of the absence of an exact four-fold degeneracy in the spectrum of the staggered Dirac operator, the fourth-root theory is non-local, and not unitary, at any non-zero lattice spacing [15]. The question that must be addressed is whether the non-locality disappears, and unitarity is recovered, in the continuum limit.

First, a degree of control over the infra-red behavior must be maintained, and I will assume that the quark masses are all positive [16, 17, 18, 19]. In a nut-shell, the following tentative reasoning summarizes how the fourth-root recipe might be valid in the continuum limit: The taste-violating effects of staggered fermions arise from irrelevant operators. These effects should scale to zero in the continuum limit like a positive power of the lattice spacing. Hence exact four-fold taste degeneracy is recovered in the continuum limit. The effective low-energy Dirac operator attains the form  $\tilde{D} \otimes \mathbf{1}$ , where  $\tilde{D}$  is a local operator that carries no taste index, and where  $\mathbf{1}$  is the four-by-four identity matrix in taste space. The fourth root of  $\det(\tilde{D} \otimes \mathbf{1})$  is  $\det(\tilde{D})$ . This fourth root is analytic, leading to locality and unitarity.<sup>3</sup>

Before making any concrete claims, a framework is needed where statements about the continuum limit will be well defined to begin with. The natural tool for the task is renormalization-group (RG) block transformations, which have already been used in the free theory [20]. A coarse-lattice theory is obtained via  $n$  blocking steps, starting from a fine-lattice theory which, in the case at hand, contains staggered fermions. The process is repeated with more and more blocking steps, but the coarse-lattice spacing (in physical units)

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<sup>1</sup> In principle, one could account for the up, down, strange, and charm quarks by the four tastes of a single staggered field with a general staggered mass term [8]. In practice, this is not done. See Ref. [11] for a discussion of the reasons.

<sup>2</sup> In the isospin limit, the up-down sector is represented by a square root of a staggered determinant with the common light quark mass.

<sup>3</sup> The analytic continuation to Minkowski space must be performed after the continuum limit.

is held fixed. With each additional blocking step the initial fine-lattice spacing gets smaller, while the bare parameters are adjusted to maintain constant physics. In the limit  $n \rightarrow \infty$ , one obtains a (well-defined!) coarse-lattice theory that generates a set of continuum-limit observables. By setting the coarse-lattice spacing to be small enough, we ensure that the observables are rich enough to extract all the QCD physics.

Using this blocking framework, I make the following two-pronged argument. I first derive the non-controversial claim that exact taste symmetry is recovered in the continuum limit of the ordinary, local staggered theory. The concrete physical properties on which this conclusion rests are itemized. Like many fundamental properties, no rigorous proofs of these physical properties are available. Yet, they are more than just plausible. It is difficult to imagine how they could be spoiled without grossly affecting the continuum limit of lattice QCD as we know it.

I next examine to what extent the same physical properties remain valid in the fourth-root theory. Properties that have to do purely with the short-distance behavior, and generalize relatively easily to the fourth-root theory, include power-counting renormalizability [11, 12, 13], and the locality of the contributions to the gauge-field effective action generated by the integration over the ultra-violet fermion modes.

A crucial ingredient is the scaling of taste-breaking effects. Because of the absence of a local fermion action, it is unclear whether a scaling analysis is at all possible. In other words, it is unclear what is the merit of the observation that the taste-breaking effects arise, formally, from an irrelevant operator. I bypass this difficulty by building a representation of the RG-blocked theory where all the taste violations can be traced back to a local operator, whose scaling can be computed by developing the appropriate perturbative expansion. I argue that the result should indeed reproduce the scaling law of an irrelevant operator.

If the taste violations vanish in the limit of infinitely many blocking steps, then, after sufficiently many blocking steps, it should be possible to find local theories in the correct universality class that provide a good approximation of the fourth-root theory. I construct such local theories explicitly via a *reweighting* of the blocked fourth-root theory that amounts to discarding all the taste-breaking terms. Starting from the reweighted theories and working back towards the fourth-root theory, I then conclude that exact taste symmetry is recovered in the continuum limit of the fourth-root staggered theory as well. As I have explained above, this implies the validity of the fourth-root recipe.

This paper is organized as follows. The RG-blocking framework is introduced in Sec. II. Technical details have been kept to a minimum, and are mostly relegated to several appendices. The beginning of Sec. II contains a brief summary both of the section itself and of the content of the appendices. Sec. II ends with the introduction of the reweighted theories, at which point I give a description of the key steps of the argument, the details of which will be presented in subsequent sections.

The recovery of exact taste symmetry in the continuum limit of the ordinary, local staggered theory is discussed in Sec. III. The reweighted theories are discussed in detail in Sec. IV. Finally, in Sec. V, the reweighted theories are used to establish the recovery of exact taste symmetry in the continuum limit of the fourth-root theory. My conclusions are given in Sec. VI.

This paper is long, and addresses both conceptual and technical questions. In order to help the reader find his/her way through, I have organized the paper such that the essentials are summarized in the following (sub)sections: Sec. II E explains the technical layout of the argument; Sec. IV D gives a summary of key properties of the reweighted theories; and

Sec. VI contains the conclusions. For a brief account of this work, see Ref. [21].

## II. THE RG BLOCKING FRAMEWORK

In this section I introduce the RG blocking framework, referring mostly to the ordinary staggered theory for pedagogical reasons. This section provides a bird’s eye overview. It contains only the minimum needed to follow the logic of the main argument, as given in subsequent sections. Much of what goes into the construction has been relegated to several appendices. In the appropriate places, I refer to the relevant appendices for a more elaborate discussion. The summary below serves as a “Table of Contents” both for this section and for the appendices.

The blocking transformations are introduced in Sec. II A which also serves to set the notation. The first blocking step is special, since it is used to make the transition from the standard one-component formalism of staggered fermions to a taste-basis representation. This special step is discussed in App. A 1. The same appendix also provides technical details on the fermion blocking kernels for subsequent blocking steps (App. A 2), and contains a proof of the positivity of fermion determinants encountered in the blocking process (App. A 3). Next, Sec. II B casts the partition function of the resulting blocked theory in a form that will be repeatedly used below. Sec. II C introduces the pull-back mapping of operators from the coarse lattice back to the original fine lattice, as well as its main uses. A more elaborated discussion of the pull-back mapping may be found in App. B, alongside with some details on the gauge-field blocking kernels. Also relegated to appendices are the generation of blocked gauge-field ensembles (App. C), a general discussion of lattice symmetries under the blocking transformation (App. D), as well as a more specific discussion of the hypercubic and chiral symmetries (App. E).

The reasons why the interacting fourth-root theory cannot be local for any finite lattice spacing have been spelled out in Ref. [15], which also contains further details on various Dirac operators encountered within the blocking framework. For completeness, a brief review of these reasons is given in App. F. The remaining two appendices (App. G and App. H) deal with scaling issues.

In this section, I continue in Sec. II D with a summary of the main lessons from the free theory [20]. Finally, I introduce the fourth-root theory and the various reweighted theories in Sec. II E, and give an overview of the argument, to be presented in detail in subsequent sections, the conclusion of which is that the fourth-root recipe is valid in the continuum limit.

### A. The blocking transformations

Originally, the partition function of the ordinary staggered theory is

$$Z = \int \mathcal{D}\mathcal{U} \mathcal{D}\chi \mathcal{D}\bar{\chi} \exp[-S_g(\mathcal{U}) - \bar{\chi} D_{stag}(\mathcal{U}) \chi], \quad (2.1)$$

where  $D_{stag}(\mathcal{U}) = D_{stag}(x, y; \mathcal{U})$  is the staggered Dirac operator, and  $\bar{\chi}(x), \chi(y)$  is the staggered field. The fine-lattice coordinates are denoted  $x, y$ , and the lattice spacing is  $a_f$ . The link variables are  $U_{\mu, x}$ , and the gauge field as a whole will be denoted  $\mathcal{U} = \{U_{\mu, x}\}$ . The gauge-field action is  $S_g(\mathcal{U})$ . Summations over all lattice sites will be suppressed.

As already mentioned, I assume that all the quark masses are positive.<sup>4</sup> In order to avoid unnecessarily cluttered notation I will consider the ordinary and fourth-root theories with a single flavor of staggered fermions. The generalization to more than one flavor is obvious.

I will perform  $n + 1$  blocking steps, labeled as  $k = 0, 1, \dots, n$ . The first,  $k = 0$  step is special; it transforms the staggered field from its usual one-component basis to a taste basis, which is then retained in all subsequent blocking steps. The  $k = 0$  step maintains the number of fermionic degrees of freedom. It is described in detail in App. A. In the subsequent blocking steps, thinning out of all degrees of freedom (fermions and gauge field) occurs. I have chosen to block (and thin out) the gauge field in the special  $k = 0$  step as well, essentially for no better reason than making the notation more tractable.

In every blocking step the lattice spacing is increased by a factor of two. Thus,  $a_k = 2^{k+1}a_f$  for  $k = 0, \dots, n$ . When speaking of the coarse-lattice theory I will refer to the theory obtained at the last,  $k = n$  step. Its lattice spacing will also be denoted  $a_c \equiv a_n$ . When I increase the number of blocking steps, the coarse-lattice spacing  $a_c$  will be held fixed in physical units, and the fine-lattice spacing will decrease as  $a_f = 2^{-n-1}a_c$ . The bare parameters on the fine lattice are adjusted to maintain constant physics. I will assume that the (fixed) length of each dimension of the coarse lattice is finite. Since, by assumption, all quarks are massive, no subtlety should arise in taking the thermodynamical limit.

For  $k = 0, \dots, n$ , the blocked fermion and anti-fermion fields on the  $k^{\text{th}}$  lattice will be denoted as  $\psi_{\alpha i}^{(k)}(\tilde{x}^{(k)})$  and  $\bar{\psi}_{\alpha i}^{(k)}(\tilde{x}^{(k)})$  respectively, where  $\tilde{x}^{(k)}$  are the coordinates on the  $k^{\text{th}}$  lattice. The indices  $\alpha$  and  $i$ , both ranging from one to four, are the Dirac and the taste index respectively. The blocked link variables will be denoted  $V_{\mu, \tilde{x}^{(k)}}^{(k)}$ . The  $k^{\text{th}}$ -step blocked gauge field as a whole is denoted  $\mathcal{V}^{(k)} = \{V_{\mu, \tilde{x}^{(k)}}^{(k)}\}$ .

Each blocking step is performed by multiplying the integrand of the partition function of the previous step by one, written in a sophisticated form [22] (for reviews of the renormalization group, see Refs. [23, 24]). Since integrations are over a compact space or else involve Grassmann variables, the order of integrations can be chosen at will. The result of the first, special blocking step, and the subsequent  $n$  ordinary blocking steps, is summarized by the following equation:

$$Z = \int \mathcal{D}\mathcal{U} \mathcal{D}\chi \mathcal{D}\bar{\chi} \exp[-S_g(\mathcal{U}) - \bar{\chi} D_{stag}(\mathcal{U}) \chi] \quad (2.2a)$$

$$\begin{aligned} & \times \int \prod_{k=0}^n \left[ \mathcal{D}\mathcal{V}^{(k)} \mathcal{D}\psi^{(k)} \mathcal{D}\bar{\psi}^{(k)} \right] \exp \left[ -\mathcal{K}(\mathcal{U}, \chi, \bar{\chi}, \{\mathcal{V}^{(k)}, \psi^{(k)}, \bar{\psi}^{(k)}\}) \right] \\ & = \int \mathcal{D}\mathcal{V}^{(n)} \mathcal{D}\psi^{(n)} \mathcal{D}\bar{\psi}^{(n)} \exp \left[ -S_n(\mathcal{V}^{(n)}, \psi^{(n)}, \bar{\psi}^{(n)}) \right], \end{aligned} \quad (2.2b)$$

where  $S_n$  is the final coarse-lattice action. In Eq. (2.2a),  $\mathcal{K}$  represents the sum of all the blocking kernels. The notation  $\{\mathcal{V}^{(k)}, \dots\}$  signifies dependence on the listed fields for all  $0 \leq k \leq n$ . Itemizing the blocking kernels,

$$\mathcal{K} = \sum_{k=0}^n \left( \mathcal{K}_B^{(k)} + \mathcal{K}_F^{(k)} \right), \quad (2.3)$$

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<sup>4</sup> See Refs. [17, 19] for a discussion of how to implement the physical theory with a negative quark-mass using the fourth-root recipe.

in which the subscripts  $B, F$  refer to bosons (*i.e.* the gauge field) and fermions respectively, we have

$$\mathcal{K}_B^{(0)} = \mathcal{B}_0(\mathcal{V}^{(0)}, \mathcal{U}) + \mathcal{N}_0(\mathcal{U}), \quad (2.4a)$$

$$\mathcal{K}_B^{(k)} = \mathcal{B}_k(\mathcal{V}^{(k)}, \mathcal{V}^{(k-1)}) + \mathcal{N}_k(\mathcal{V}^{(k-1)}), \quad (2.4b)$$

where

$$\exp[+\mathcal{N}_0(\mathcal{U})] = \int \mathcal{D}\mathcal{V}^{(0)} \exp\left[-\mathcal{B}_0(\mathcal{V}^{(0)}, \mathcal{U})\right], \quad (2.5a)$$

$$\exp\left[+\mathcal{N}_k(\mathcal{V}^{(k-1)})\right] = \int \mathcal{D}\mathcal{V}^{(k)} \exp\left[-\mathcal{B}_k(\mathcal{V}^{(k)}, \mathcal{V}^{(k-1)})\right], \quad (2.5b)$$

and

$$\mathcal{K}_F^{(0)} = \alpha_0 \left( \overline{\psi}^{(0)} - \overline{\chi} Q^{(0)\dagger}(\mathcal{U}) \right) \left( \psi^{(0)} - Q^{(0)}(\mathcal{U}) \chi \right), \quad (2.6a)$$

$$\mathcal{K}_F^{(k)} = \alpha_k \left[ \overline{\psi}^{(k)} - \overline{\psi}^{(k-1)} Q^{(k)\dagger}(\mathcal{V}^{(k-1)}) \right] \left[ \psi^{(k)} - Q^{(k)}(\mathcal{V}^{(k-1)}) \psi^{(k-1)} \right]. \quad (2.6b)$$

In Eqs. (2.4b), (2.5b) and (2.6b), the  $k$ -range is  $1 \leq k \leq n$ . The fermion blocking kernel  $Q^{(k)}(\mathcal{V}^{(k-1)})$  depends on the gauge field on the  $(k-1)^{\text{th}}$  lattice only. It is an ultra-local and gauge covariant linear transformation that maps each  $2^4$  hypercube of the  $(k-1)^{\text{th}}$  lattice to a single site of the  $k^{\text{th}}$  lattice. The blocking parameter  $\alpha_k$  has mass-dimension one. In this paper I will assume that  $\alpha_k$  is chosen to be of order  $a_k^{-1} = 2^{n-k} a_c^{-1}$ . Note that the gaussian integral

$$\int \mathcal{D}\psi^{(k)} \mathcal{D}\overline{\psi}^{(k)} \exp\left(-\mathcal{K}_F^{(k)}\right), \quad (2.7)$$

yields a trivial constant; in Eq. (2.2a), this constant was absorbed into the definition of the Grassmann measure. For more details on the fermion blocking kernels, see App. A. For the gauge-boson blocking kernels, see App. B.

## B. A convenient representation

Keeping track of the taste-symmetry violations is evidently important. For this purpose, the action  $S_n$  that results from the blocking transformations is not very useful because it contains multi-fermion interactions. In the fourth-root theory, that action would furthermore be non-local, and completely intractable. The problem can be circumvented by noting that the fermion blocking transformations are gaussian, as can be seen from Eqs. (2.2a) and (2.6). Let us integrate out the original fermion variables as well as all the blocked fermion variables, except for those that live on the coarsest lattice. But let us not integrate out the original gauge field nor any of the blocked gauge fields. All the integrations we explicitly do are, thus, gaussian. The result is

$$Z = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n \left[ \mathcal{D}\mathcal{V}^{(k)} \right] \mathcal{B}_n\left(1; \mathcal{U}, \{\mathcal{V}^{(k)}\}\right) \int \mathcal{D}\psi^{(n)} \mathcal{D}\overline{\psi}^{(n)} \exp\left(-\overline{\psi}^{(n)} D_n \psi^{(n)}\right), \quad (2.8)$$

where

$$\begin{aligned} B_n(\gamma; \mathcal{U}, \{\mathcal{V}^{(k)}\}) &= \exp\left(-S_g - \sum_{k=0}^n \mathcal{K}_B^{(k)}\right) \\ &\times \det^\gamma\left[\left(\alpha_0 G_0\right)^{-1}\right] \prod_{k=1}^n \det^\gamma\left[\left(\alpha_k^{1/16} G_k\right)^{-1}\right]. \end{aligned} \quad (2.9)$$

In the ordinary staggered theory one has  $\gamma = 1$ . In the fourth-root theory we will have  $\gamma = \frac{1}{4}$ . Here

$$D_0^{-1} = \alpha_0^{-1} + Q^{(0)} D_{stag}^{-1} Q^{(0)\dagger}, \quad (2.10a)$$

$$D_k^{-1} = \alpha_k^{-1} + Q^{(k)} D_{k-1}^{-1} Q^{(k)\dagger}, \quad k = 1, \dots, n, \quad (2.10b)$$

$$G_0^{-1} = D_{stag} + \alpha_0 Q^{(0)\dagger} Q^{(0)}, \quad (2.11a)$$

$$G_k^{-1} = D_{k-1} + \alpha_k Q^{(k)\dagger} Q^{(k)}. \quad k = 1, \dots, n. \quad (2.11b)$$

Equation (2.9) includes the constant resulting from the Grassmann integration (2.7), for each  $k$ . The different powers of  $\alpha_k$  arise because the  $(k+1)^{\text{th}}$  lattice has sixteen times fewer fermionic degrees of freedom compared to the  $k^{\text{th}}$  lattice, except for the  $k=0$  step which does not thin out the number of fermionic degrees of freedom.

For fixed values of the original as well as all the blocked gauge fields, we have the factorization formula [22]

$$\det(D_{stag}) = \det(D_n) \det\left[\left(\alpha_0 G_0\right)^{-1}\right] \prod_{k=1}^n \det\left[\left(\alpha_k^{1/16} G_k\right)^{-1}\right]. \quad (2.12)$$

Equation (2.12) shows how the fermionic short-distance fluctuations are gradually removed from the theory. Each factor of  $\det(G_k^{-1})$  results from integrating out fermionic degrees of freedom during the  $k$ -th blocking step, and generates an effective action for the collection of gauge fields  $\mathcal{U}, \mathcal{V}^{(0)}, \dots, \mathcal{V}^{(k-1)}$ ,

$$S_{eff}^0 = \log \det(\alpha_0 G_0), \quad (2.13a)$$

$$S_{eff}^k = \log \det\left(\alpha_k^{1/16} G_k\right), \quad k = 1, \dots, n. \quad (2.13b)$$

All the long-distance physics is contained in the RG-blocked Dirac operator  $D_n$ . That  $D_n$  faithfully reproduces the long-distance physics can be seen as follows. By successive applications of Eq. (2.10), the blocked-field propagator  $D_n^{-1}$  can be expressed in terms of the original propagator  $D_{stag}^{-1}$  between a special type of smeared sources built up from the product of the fermion blocking kernels (cf. Eq. (A6)). As I explain below, this is but a special case of a more general mechanism.

For  $m > 0$ , one can show that  $\det(D_k)$  is strictly positive and  $\det(G_k^{-1})$  is positive. See App. A 3 for the proof.

### C. Pull-back mapping

The RG-blocking transformations start off at the cutoff scale and proceed gradually towards lower energy scales. But the blocking transformations facilitate another operation

that works in the reverse direction. Suppose that we want to calculate the expectation value of an operator  $\mathcal{O}^{(n)} = \mathcal{O}^{(n)}(\mathcal{V}^{(n)}, \psi^{(n)}, \bar{\psi}^{(n)})$  defined explicitly in terms of the fields of the  $n^{\text{th}}$  lattice. Using Eq. (2.2), the expectation value may be calculated as follows. We begin by integrating over the  $n^{\text{th}}$ -lattice fields, then over the  $(n-1)^{\text{th}}$ -lattice fields, and so on. If, however, we stop at any intermediate step  $k$ , the result of the integrations we have done so far will be expressible solely in terms of the fields of the  $k^{\text{th}}$  lattice. This procedure defines a *pull-back* mapping of any operator from the  $n^{\text{th}}$  to the  $k^{\text{th}}$  lattice, that by construction preserves expectation values.

Explicitly, for any  $-1 \leq j < n-1$ , the pull-back mapping  $\mathcal{T}^{(j,n)} : \mathcal{O}^{(n)} \rightarrow \mathcal{O}^{(j)}$  is defined by

$$\mathcal{T}^{(j,n)} \mathcal{O}^{(n)} = \int \prod_{k=j+1}^n \left[ \mathcal{D}\mathcal{V}^{(k)} \mathcal{D}\psi^{(k)} \mathcal{D}\bar{\psi}^{(k)} \right] \exp \left[ - \sum_{k=j+1}^n \left( \mathcal{K}_B^{(k)} + \mathcal{K}_F^{(k)} \right) \right] \mathcal{O}^{(n)}. \quad (2.14)$$

As promised, by construction the pull-back mapping preserves the value of observables,

$$\langle \mathcal{T}^{(j,n)} \mathcal{O}^{(n)} \rangle_j = \langle \mathcal{O}^{(n)} \rangle_n. \quad (2.15)$$

Here the expectation value  $\langle \cdots \rangle_n$  is defined by the representation of the partition function in Eq. (2.2a). Taken together, these equations merely say that we may perform the integrations in Eq. (2.2) by first integrating over the blocked fields labeled by  $j+1 \leq k \leq n$ , and then integrating over the remaining blocked fields as well as over the original fields. The value  $j = -1$  accounts for the original fine-lattice theory, and  $\mathcal{T}^{(-1,n)}$  is the pull-back from the last-step coarse lattice all the way to the original staggered theory on the fine lattice.

The pull-back mapping is ultra-local if and only if the blocking kernels are. An operator supported on a compact subset of the  $n^{\text{th}}$  lattice is mapped by  $\mathcal{T}^{(j,n)}$  to an operator supported on a corresponding, only somewhat bigger, subset of the  $j^{\text{th}}$  lattice.

An immediate corollary is that the coarse-lattice observables form a proper subset of the fine-lattice observables. The coarse-lattice expectation value of the operator  $\mathcal{O}^{(n)}$  is equal to the fine-lattice expectation value of the operator  $\mathcal{T}^{(-1,n)} \mathcal{O}^{(n)}$ . As alluded to earlier, the reconstruction of the blocked fermion propagator  $D_n^{-1}$  from its predecessors is in fact an example of the pull-back mapping in action.

This innocuous corollary leads to another, all important, result. Every coarse-lattice observable, being simultaneously a fine-lattice observable via the pull-back mapping, is constrained by all the fine-lattice symmetries. In this sense, the (physical) consequences of the exact lattice symmetries cannot “be lost” by the blocking process.

More can be said on the role of specific lattice symmetries within the blocking framework. The interested reader is referred to App. B for a more detailed discussion of the pull-back mapping. Effects of the blocking transformations on the fine-lattice symmetries, including the relevance of the pull-back mapping, are discussed in App. D and App. E.

## D. Lessons from the free theory

In this subsection I review the main results of RG-blocking in the free theory [20]. With no gauge fields, Eq. (2.12) takes a somewhat special form. By a unitary change of variables one can switch back and forth between the Dirac operators  $D_{stag}$ , in the one-component



basis, and  $D_{taste}$ , in the taste basis [5, 6], and<sup>5</sup>

$$\det(D_{stag}) = \det(D_{taste}) = \det(D_n) \prod_{k=1}^n \det \left[ \left( \alpha_k^{1/16} G_k \right)^{-1} \right]. \quad (2.16)$$

An explicit expression for the free RG-blocked Dirac operator  $D_n$  may be written down. Its taste-violating part  $\Delta_n$  (see Eq. (2.21) below) has a norm bounded by<sup>6</sup>

$$\|a_c \Delta_n\| = O(2^{-n}) = O(a_0/a_c). \quad (2.17)$$

In the limit  $n \rightarrow \infty$ , all the taste-violating terms go to zero, and

$$\lim_{n \rightarrow \infty} D_n = D_{rg} \otimes \mathbf{1}, \quad (2.18a)$$

$$\lim_{n \rightarrow \infty} \frac{\det(D_{taste})}{\prod_{k=1}^n \det(G_k^{-1})} = \det^4(D_{rg}). \quad (2.18b)$$

Again,  $\mathbf{1}$  is the identity matrix in taste space. The “one-taste” operator  $D_{rg}$  is local, and  $\det(D_{rg})$  qualifies as a fourth root of  $\det(D_{taste}) = \det(D_{stag})$  in the sense of Eq. (2.18b).

By repeatedly integrating out the short-distance fluctuations we thus obtain a coarse-lattice operator with an exact four-fold degeneracy in the limit  $n \rightarrow \infty$ . The power-law scaling of the taste-breaking terms is clearly as dictated by their origin: irrelevant operators with mass-dimension equal to five. Intuitively this can be understood as follows. The dimension-five taste-violating terms in  $D_{taste}$  are multiplied by an explicit factor of  $a_0$ , the initial taste-basis lattice spacing. But the momentum flowing through the fermion line is in effect of order  $|p| \sim a_c^{-1}$  at most. The relative size of the taste-violating terms is therefore at most of order  $a_0/a_c$ .

As a simple corollary of the rigorous work of Ref. [25], one can prove that  $D_k$ ,  $G_k^{-1}$ , and its inverse  $G_k$ , are all local, bounded operators. Mathematical rigor set aside, one can understand how  $G_k^{-1}$  develops an  $O(\alpha_k)$  gap directly from Eq. (2.11). Since the fermion mass is very small in any lattice units, it can be ignored for this purpose. Also, from now on, I assume that  $\alpha_0$  in Eq. (2.10a) has a finite,  $O(a_0^{-1})$  value. Hence Eq. (2.12) (and not Eq. (2.16)) must be used also in the free theory. For further details on the  $k = 0$  step see Ref. [15] and App. A.

For  $k \geq 1$ , the massless Dirac operator  $D_{k-1}$  satisfies a Ginsparg-Wilson (GW) relation [26], in which  $[\gamma_5 \otimes \xi_5]$  takes the role usually played by  $\gamma_5$ . Here  $\xi_5$  is the representation of  $\gamma_5$  that acts on the taste index (see App. A).<sup>7</sup> The eigenvalues of  $D_{k-1}$  thus lie on a circle in the right half of the complex plane, with the imaginary axis tangent to the eigenvalue circle on the left [15, 20]. In order to obtain  $G_k^{-1}$  from  $D_{k-1}$ , we add the blocking-kernel term  $\alpha_k Q^{(k)\dagger} Q^{(k)}$ . This new term is positive semi-definite. It affects mostly the small-momentum

<sup>5</sup> Equation (2.12) reduces to Eq. (2.16) in the limit  $\alpha_0 \rightarrow \infty$ , where  $\det((\alpha_0 G_0)^{-1}) \rightarrow 1$  and  $D_0 \rightarrow D_{taste}$  [15]. What I denote as  $G_k$  in this paper was denoted  $\Gamma_k$  in Ref. [20], see Eq. (6) therein. The notation  $G_k$  has a related, but different meaning in Ref. [20]. Also, for compatibility with Ref. [25], in Ref. [20] the blocking parameters  $\alpha_k$  were all chosen to have a fixed,  $O(1)$  value in units of the coarse-lattice spacing, whereas here I make the more natural assumption  $\alpha_k = O(a_k^{-1})$ .

<sup>6</sup> This bound is rigorous in the free theory [20, 25].

<sup>7</sup> In the massless limit  $D_{rg}$  (Eq. (2.18)) satisfies the usual Ginsparg-Wilson relation.

modes located near the origin of the Brillouin zone, and pushes their eigenvalues to the right by an amount proportional to  $\alpha_k$ . The result is that no eigenvalue remains in an  $O(\alpha_k)$  neighborhood of the origin. In other words,  $G_k^{-1}$  has developed a gap of order  $\alpha_k$ .<sup>8</sup> Its inverse  $G_k$  will thus have a decay rate of order  $\alpha_k$  as well. Furthermore, since one may also obtain the blocked Dirac operator as

$$D_k = \alpha_k - \alpha_k^2 Q^{(k)} G_k Q^{(k)\dagger}, \quad (2.19)$$

it follows that the decay rates of the kernels of  $D_k$  and of  $G_{k+1}^{-1}$  should be  $O(\alpha_k)$  too. The argument may now be repeated for the  $(k+1)^{\text{th}}$  step.

### E. Overview of the argument

When no roots are taken, the local lattice theory with  $N_f$  staggered fields belongs to the *universality class* of QCD with  $4N_f$  quarks. The usual universality classification is, however, inapplicable to the fourth-root theory, because of its non-locality. There is evidence from staggered chiral perturbation theory that the outcome of taking the fourth root may be described in terms of an extended Hilbert space containing unphysical states with non-zero taste charges: the contributions of these taste-charged states as intermediate states are thus also unphysical. A unitary subspace with the correct number of quarks, one per staggered field, will exist only in the continuum limit, and only provided that exact taste symmetry is recovered [15, 19, 27, 28].

The range of the non-locality present in the fourth-root theory appears to be set by infra-red scales of the theory: the masses of the various staggered pions [15, 21]. The taste-breaking terms driving the non-locality are lattice artifacts, and would naively be expected to vanish in the continuum limit. If indeed exact taste symmetry is recovered in the continuum limit of the fourth-root theory, then, when the lattice spacing has become small enough, it is logically necessary that local lattice theories in the desired universality class exist which provide a good approximation of the fourth-root theory. Such local theories could be constructed by simply discarding the taste-violating terms from the blocked fourth-root theory. This is the idea behind the introduction of *reweighted* theories.

Consider first the ordinary, local staggered theory. With the help of Eqs. (2.8) and (2.9), the original staggered partition function in Eq. (2.1) can be re-expressed as

$$Z = Z_n \equiv \int \mathcal{D}\mathcal{U} \prod_{k=0}^n \left[ \mathcal{D}\mathcal{V}^{(k)} \right] B_n \left( 1; \mathcal{U}, \{ \mathcal{V}^{(k)} \} \right) \det(D_n). \quad (2.20)$$

In short, this expression results from  $n+1$  blocking steps, after which the fermion fields have been integrated out altogether, while retaining explicitly the integral over the original, fine-lattice gauge field  $\mathcal{U}$ , as well as over the gauge fields  $\mathcal{V}^{(k)}$  of all the blocking steps. Recall that the special  $k=0$  step facilitates the transition from the usual one-component staggered basis to a taste basis.

In order to keep track of taste-symmetry violations, let us split the blocked Dirac operator

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<sup>8</sup> When the eigenvalues  $\lambda_i$  are complex we may define the gap as  $\min |\lambda_i|$ .

$D_n$  into its taste-invariant and non-invariant parts:

$$D_n = D_{inv,n} + \Delta_n, \quad (2.21a)$$

$$D_{inv,n} = \tilde{D}_{inv,n} \otimes \mathbf{1}, \quad (2.21b)$$

$$\tilde{D}_{inv,n} = \frac{1}{4} \text{tr}_{ts}(D_n), \quad (2.21c)$$

where  $\text{tr}_{ts}$  denotes tracing over the taste index only. The taste non-invariant part  $\Delta_n$  is traceless on the taste index.

By construction,  $D_n$  accounts for physics over distances on the order of the coarse-lattice spacing  $a_c$  or longer. In particular, its taste violating part  $\Delta_n$  accounts for all taste-symmetry violations at the energy scale  $a_c^{-1}$  and below. The question is how big are the taste-symmetry violating effects in the spectrum of  $D_n$ .

Originally, the staggered Dirac operator  $D_{stag}$  exhibits taste-symmetry violations on all scales. In the low-lying spectrum they are small [29]; but they grow gradually with the energy scale, until they become  $O(1)$  at the scale of the original lattice cutoff  $a_f^{-1} = 2^{n+1}a_c^{-1}$ . I will argue that, nevertheless, by choosing  $n$  large enough the taste-violating effects in the spectrum of  $D_n$  can be made arbitrarily small. RG blocking removes the ultra-violet fermionic modes. Their remnant is, *mutatis mutandis*, the effective action  $S_{eff}^k$  (Eq. (2.13)). This effective action is a sum of (products of) Wilson loops and, I will claim, it is local on both the ordinary and fourth-root staggered ensembles. Being a local functional of the (original and blocked) gauge fields, but not a functional of the fermion fields, it cannot give rise to any taste-symmetry violations at large distances. In this sense, the ultra-violet taste violations have been eliminated. I will further claim that, with every additional blocking step, the remaining taste-violations in the entire eigenvalue spectrum of the blocked Dirac operator get smaller, uniformly, basically because this spectrum consists of only “low-energy” modes with respect to the fine-lattice scale. In the limit of infinitely many blocking steps, taste symmetry is fully recovered.

We may discard all the taste-breaking effects from the staggered theory, by hand, after only a finite number of blocking steps. Truncating the blocked Dirac operator in Eq. (2.20) to its taste-invariant part  $D_{inv,n}$  gives rise to the following reweighted theory

$$Z_{inv,n} = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] B_n(1; \mathcal{U}, \{\mathcal{V}^{(k)}\}) \det(D_{inv,n}). \quad (2.22a)$$

A more conventional looking path-integral representation may be obtained by rewriting  $\det(D_{inv,n})$  as a path integral over (four-taste) coarse-lattice fermion fields  $\psi^{(n)}, \bar{\psi}^{(n)}$ , as in Eq. (2.8), and then integrating out the “tower” of gauge fields except for the coarse-lattice gauge field  $\mathcal{V}^{(n)}$ . This gives

$$Z_{inv,n} = \int \mathcal{D}\mathcal{V}^{(n)} \mathcal{D}\psi^{(n)} \mathcal{D}\bar{\psi}^{(n)} \exp \left[ - S_{inv,n}(\mathcal{V}^{(n)}, \psi^{(n)}, \bar{\psi}^{(n)}) \right], \quad (2.22b)$$

which is to be compared with the path integral representation (2.2b) of the blocked staggered theory. Unlike the staggered theory, the reweighted theory has no shift invariance [7, 8]. Instead, it has exact taste- $U(4)$  invariance by construction. Another difference is that the above constructed reweighted theory does not have an exact chiral symmetry in the massless

limit.<sup>9</sup>

Reweighting at blocking level  $n$  generates a sequence of theories  $Z_{inv,n}$  which are different from each other, as well as from the staggered theory. But I will argue in Sec. III that, because  $\Delta_n$  is an irrelevant operator, the (sequence of) reweighted theories has the same continuum limit as the (blocked) staggered theory. Each reweighted theory enjoys exact taste symmetry by construction, and this implies (the uncontroversial result) that exact taste symmetry is recovered in the continuum limit of the ordinary staggered theory. The proof works by establishing the existence of a convergent expansion relating the staggered and reweighted theories when  $n$  is large enough. One must require that all the quark masses be non-zero, consistent with the fact that the chiral and continuum limits do not always commute [16, 17, 18, 19].

Moving on to the fourth-root theory, its partition function cannot be represented as an ordinary path integral with a local fermion action. Rather, it is given by [10]

$$Z^{root} = \int \mathcal{D}\mathcal{U} \exp(-S_g) \det^{1/4}(D_{stag}), \quad (2.23)$$

where the positive fourth root is taken. As in the ordinary staggered theory, this may be re-expressed in an  $n$ -step RG-blocked form as

$$Z^{root} = Z_n^{root} \equiv \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] B_n\left(\frac{1}{4}; \mathcal{U}, \{\mathcal{V}^{(k)}\}\right) \det^{1/4}(D_n). \quad (2.24)$$

Again let us remove the taste-breaking terms by hand, which gives rise to a new family of reweighted theories

$$Z_{inv,n}^{root} = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] B_n\left(\frac{1}{4}; \mathcal{U}, \{\mathcal{V}^{(k)}\}\right) \det(\tilde{D}_{inv,n}). \quad (2.25)$$

Here I have used the exact taste symmetry of  $D_{inv,n} = \tilde{D}_{inv,n} \otimes \mathbf{1}$  to take the analytic fourth root of its determinant.<sup>10</sup>

One can represent  $\det(\tilde{D}_{inv,n})$  as a fermion path integral. This suggests that the validity of the continuum limit of the fourth-root theory could be established by, once again, showing that the sequence of reweighted theories has the same continuum limit as the blocked fourth-root theory. But we must now face two hurdles that were not encountered in the local, ordinary staggered theory.

The new hurdles are addressed in Sec. IV. First, we must show that the reweighted theories derived from the fourth-root theory are local. This is done in Sec. IV B by showing that, on the basis of plausible assumptions, the effective action  $S_{eff}^k$  and the blocked Dirac operator  $D_k$  are local on the  $k^{\text{th}}$  lattice scale, on both the ordinary and the fourth-root staggered ensembles. Because  $\tilde{D}_{inv,n}$  is defined by a trace projection,  $\tilde{D}_{inv,n}$  and  $\Delta_n$  are then separately local.

Introducing coarse-lattice Dirac fields  $q^{(n)}, \bar{q}^{(n)}$  that, this time, carry no taste index, and once again integrating out the “tower” of gauge fields except for the coarse-lattice gauge field

<sup>9</sup> More sophisticated reweighted theories may be constructed. See Ref. [15] for a construction that maintains the exact chiral symmetry of the  $m \rightarrow 0$  limit.

<sup>10</sup> For large enough  $n$ ,  $\det(\tilde{D}_{inv,n})$  and  $\det(\tilde{D}_{inv,n})$  are positive, see Sec. V.

$\mathcal{V}^{(n)}$  the reweighted fourth-root partition function then takes the form (compare Eq. (2.22b)),

$$Z_{inv,n}^{root} = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n \left[ \mathcal{D}\mathcal{V}^{(k)} \right] B_n\left(\frac{1}{4}; \mathcal{U}, \{\mathcal{V}^{(k)}\}\right) \quad (2.26a)$$

$$\times \int \mathcal{D}q^{(n)} \mathcal{D}\bar{q}^{(n)} \exp\left(-\bar{q}^{(n)} \tilde{D}_{inv,n} q^{(n)}\right) \\ = \int \mathcal{D}\mathcal{V}^{(n)} \mathcal{D}q^{(n)} \mathcal{D}\bar{q}^{(n)} \exp\left[-S_{inv,n}^{root}\left(\mathcal{V}^{(n)}, q^{(n)}, \bar{q}^{(n)}\right)\right]. \quad (2.26b)$$

The “one-taste” action  $S_{inv,n}^{root} = S_{inv,n}^{root}(\mathcal{V}^{(n)}, q^{(n)}, \bar{q}^{(n)})$  is complicated, and contains many multi-fermion interactions, just like  $S_n$  and  $S_{inv,n}$  encountered earlier. What matters, however, is that  $S_{inv,n}^{root}$  too is local on the coarse-lattice scale if, in particular,  $\tilde{D}_{inv,n}$  is; the “surgery” of removing the taste violations has also removed the non-localities of the blocked fourth-root theory!

I stress that the argument for locality of  $S_{eff}^k$  and  $D_k$  does *not* require that the underlying theory be local. This non-perturbative argument is very general, and only makes mild use of the *renormalizability* of all theories including the fourth-root theory [11, 12] to establish the existence of a weak-coupling regime.

At this point we expect that the reweighted theories  $Z_{inv,n}^{root}$  derived from the fourth-root theory are local, and belong to the desired universality class. Now comes the second hurdle. Convergence of the reweighted and the staggered theories to the same continuum limit depends on the scaling of the taste-breaking effect. But the fourth-root theory does not have a local action in the first place, so how are we to perform any scaling analysis?

The scaling of taste-breaking effects in gauge-invariant observables can, of course, be studied numerically, and what one finds is in agreement with what one would naively expect [9, 28, 30]. But, while no undesirable effects have been encountered at presently accessible values of the lattice spacing and the quark masses, numerical results alone cannot alleviate the concern that closer to the continuum and/or the chiral limit the scaling of taste-breaking effects might eventually be altered in undesirable ways due to the lack of a local lattice action.

On the lattice, a scaling analysis rests on two pillars. On the non-perturbative side, we define a theory, construct local operators within it, and set up an RG transformation. On the perturbative side, if the theory is power-counting renormalizable, we can compute the scaling of any local operator. An important special case is to take the local operator to be the action itself, or individual terms within it.

In this paper I show how to generalize the scaling analysis to the fourth-root theory. First, the fourth-root theory *is* renormalizable. The multi-gauge-field representation of the blocked fourth-root theory, Eq. (2.24), then allows us to bypass the lack of a local fermion action. Instead, we may study the scaling of the blocked Dirac operator  $D_k$  and the effective action  $S_{eff}^k$ . Having first shown by non-perturbative considerations that both of them are local operators, their scaling can be computed by setting up the appropriate perturbative expansion, which is done in Sec. IVC. I find that any local operator scales in the same way in the staggered theory and in the corresponding reweighted theories (with or without the fourth root). In particular, the taste-breaking part of the blocked Dirac operator,  $\Delta_n$ , indeed scales as an irrelevant operator.

Finally, in Sec. V I reconstruct the rooted theory from the corresponding reweighted theories, and establish the validity of its continuum limit.

My conclusion rests on renormalizability of the fourth-root theory, concerning which there is little doubt, and the fact that renormalizability is “inherited” by the reweighted theories (Sec. IV A). My conclusion also rests on two additional key features that have to do with locality (Sec. IV B), and scaling (Sec. IV C). I give plausible arguments for each of them, but confirmation must await more detailed future investigations. Where, then, do we stand today? One can draw an indirect but important lesson from the ordinary staggered theory. In that case we (believe we) know what is the continuum limit. Moreover, the reweighted theories are tightly constrained by the convergent expansion relating them to the – local – staggered theory. This leaves little doubt that all the key properties are valid in this case. But the argumentation of Sec. IV makes very little discrimination between the ordinary and the fourth-root cases. As I explain in more detail later, this increases confidence that nothing essential has been overlooked, and that the claimed properties are valid in the fourth-root case as well.

### III. CONTINUUM LIMIT OF THE ORDINARY STAGGERED THEORY

In this section I discuss the continuum limit of the ordinary staggered-fermion theory. Continuum-limit observables that can be computed within the coarse-lattice theory are introduced in Sec. III A. In Sec. III B, I list scaling relations that follow from a standard RG analysis in the ordinary staggered theory. These scaling relations imply the recovery of taste symmetry in the continuum limit. This is inferred in Sec. III C by comparing the blocked staggered theory to the reweighted theory at each blocking level  $n$ . Provided that the renormalized quark mass is non-zero, I show that the two theories are connected by a convergent expansion when  $n$  is large enough, and that any difference between them vanishes in the limit  $n \rightarrow \infty$ .

#### A. Continuum-limit observables

The continuum limit corresponds to the limit of infinitely many blocking steps, which is taken while holding fixed the coarse-lattice spacing  $a_c$  in physical units. The fine-lattice spacing  $a_f$  of the staggered theory goes to zero,  $a_f \Lambda = 2^{-n-1} a_c \Lambda \rightarrow 0$ , where  $\Lambda$  is the QCD scale. Constant physics is maintained by adjusting the bare parameters such that the renormalized gauge coupling  $g_r(a_c)$  and quark mass  $m_r(a_c)$  are kept fixed. The coarse-lattice spacing plays the role of the renormalization scale. The fermion’s wave-function renormalization is controlled by the parameter  $z^{(k)}$  in Eq. (A5), that fixes the overall normalization of the gauge-covariant blocking kernel  $Q^{(k)}$  in the interacting theory. I will assume that the  $z^{(k)}$ ’s have been adjusted so as to impose a wave-function renormalization condition on the blocked fermion fields at the renormalization scale  $a_c$ .

For simplicity I will restrict the discussion to meson observables. Sources for mesons are added into the fermion action on the  $n^{\text{th}}$  lattice as follows:<sup>11</sup>

$$S_{\text{source}}^{(n)}(J) = \overline{\psi}^{(n)} J \cdot \mathcal{S}^{(n)} \psi^{(n)}, \quad (3.1)$$

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<sup>11</sup> For general sources, see appendix B of Ref. [11].

where

$$\bar{\psi}^{(n)} J \cdot \mathcal{S}^{(n)} \psi^{(n)} \equiv \sum_{\tilde{x}^{(n)}} \sum_i J_i(\tilde{x}^{(n)}) O_i^{(n)}(\tilde{x}^{(n)}), \quad (3.2)$$

$$\mathcal{O}_i^{(n)}(\tilde{x}^{(n)}) = \sum_{\tilde{y}^{(n)}, \tilde{z}^{(n)}} \bar{\psi}^{(n)}(\tilde{y}^{(n)}) \mathcal{S}_i^{(n)}(\tilde{x}^{(n)}, \tilde{y}^{(n)}, \tilde{z}^{(n)}; \mathcal{V}^{(n)}) \psi^{(n)}(\tilde{z}^{(n)}). \quad (3.3)$$

The kernels  $\mathcal{S}_i^{(n)}$  are gauge-covariant and ultra-local.<sup>12</sup> Augmenting the fermion action in Eq. (2.8) by the source term (3.2) and performing the Grassmann integration we obtain the partition function with sources (compare Eq. (2.20)):

$$Z_n(a_c; J) = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] \mathcal{B}_n(1; \mathcal{U}, \{\mathcal{V}^{(k)}\}) \det(D_n + J \cdot \mathcal{S}^{(n)}), \quad (3.4)$$

as well as the normalized version

$$\mathcal{Z}_n(a_c; J) = Z_n(a_c; J) / Z_n(a_c; 0). \quad (3.5)$$

Meson correlation functions, renormalized at the scale  $a_c$ , are generated by functional differentiation of  $\mathcal{Z}_n(a_c; J)$ . The (assumed) existence of the continuum limit means that the  $n \rightarrow \infty$  limit of the normalized generating functional  $\mathcal{Z}_n(a_c; J)$  is smooth:

$$\mathcal{Z}_\infty(a_c; J) = \lim_{n \rightarrow \infty} \mathcal{Z}_n(a_c; J). \quad (3.6)$$

By differentiation of  $\mathcal{Z}_\infty(a_c; J)$  one generates continuum-limit meson correlators in euclidean space.

Before moving on let me comment on the set of coarse-lattice observables. In Eq. (2.15), which states the equality of observables under the pull-back mapping, let us choose  $j = n-1$ . Instead of using the number of blocking steps as a label, we may use the corresponding lattice spacing for this purpose. The equation then takes the form

$$\langle \mathcal{T}(a_c/2, a_c) \mathcal{O}(a_c) \rangle_{a_c/2} = \langle \mathcal{O}(a_c) \rangle_{a_c}, \quad (3.7)$$

in self-explanatory notation. Equation (3.7) remains valid in the continuum limit, where the pull-back mapping  $\mathcal{T}(a_c/2, a_c)$  remains well defined. This equation identifies the observables of the coarse-lattice theory  $\mathcal{Z}_\infty(a_c; J)$  with a proper subset of the observables of  $\mathcal{Z}_\infty(a_c/2; J)$ , the coarse-lattice theory with half the lattice spacing. Because observables are lost in the blocking, we need to take  $a_c \Lambda$  to be small enough in the first place to ensure that the observables derived from  $\mathcal{Z}_\infty(a_c; J)$  are rich enough to extract all the QCD physics. Additional reasons for choosing  $a_c \Lambda$  small will be encountered in Sec. IV.

## B. Scaling of irrelevant operators

I now list the scaling laws needed to establish the recovery of exact taste symmetry in the continuum limit of the ordinary staggered theory, as they apply within the RG-blocking framework of this paper.

<sup>12</sup> See App. A 2 for the renormalization of composite coarse-lattice operators.

First, the very existence of the continuum limit in QCD derives from asymptotic freedom,<sup>13</sup> or, in other words, from the scaling properties of the running coupling  $g_r(a_c)$  as a function of the bare coupling.

The remaining scaling laws pertain to the fermion sector. For the restoration of taste symmetry in all observables, two scaling laws will be necessary:

$$\|D_n^{-1}\| \lesssim \frac{1}{m_r(a_c)}, \quad (3.8)$$

$$\|\Delta_n\| \lesssim \frac{a_f}{a_c^2} = \frac{2^{-(n+1)}}{a_c}. \quad (3.9)$$

These scaling laws are assumed to hold in an *ensemble average* sense; they do not hold on all configurations, but they (are assumed to) hold after averaging over the configurations in the ensemble. Each configuration is generated as described in App. C: one complete configuration consists of a “mother” configuration  $\mathcal{U}_i$  of the fine-lattice gauge field, as well as of “daughter” configurations  $\mathcal{V}_i^{(1)}, \dots, \mathcal{V}_i^{(n)}$  of blocked gauge fields.

The bound (3.8) relates the lowest eigenvalues of the blocked Dirac operator  $D_n$  to the running quark mass  $m_r(a_c)$ . This inequality says that the value of  $m_r(a_c)$  measured on an ensemble of configurations is set by the lowest eigenvalues of  $D_n$ , provided that the wave-function renormalization condition was imposed at the coarse-lattice scale. The bound (3.8) is needed to tame the infra-red behavior in the fermion sector. We will see in the next subsection how it enters. For some further comments on the bound (3.8) and its parallel in the fourth-root theory, see App. G.

The crucial scaling law (3.9) determines the scaling of  $\Delta_n$ , the taste-breaking part of the blocked Dirac operator  $D_n$  (see Eq. (2.21)). In the taste representation, the leading taste violations arise from dimension-five irrelevant operators [5, 6, 20], and the right-hand side of Eq. (3.9) gives the anticipated scaling based on this engineering dimension (compare Eq. (2.17) for the free theory). The “ $\lesssim$ ” sign means that in the interacting theory the inequality holds up to logarithmic corrections, that is, powers of  $\log(a_c/a_f)$  or, equivalently, powers of  $n$ .

In a standard RG application the fermions and the gauge field are both blocked at each step, and the scaling laws apply to (the parameters of) the blocked action. Here, the scaling laws are assumed within the representation (2.8), which is superficially different in that none of the gauge fields have been integrated over explicitly. However, because the sources couple to coarse-lattice fields only (cf. Eq. (3.3)), we may imagine that we first integrate over  $\mathcal{U}, \mathcal{V}^{(1)}, \dots, \mathcal{V}^{(n-1)}$  and only later over the coarse-lattice (fermions and) gauge field. This is equivalent to inserting the source term into Eq. (2.2b) which contains the blocked-lattice action  $S_n$ . Thus, in the context of the ordinary staggered theory, the above scaling laws are on the same footing as the scaling laws used in a completely standard RG-blocking context.

Because of the scaling of the taste-breaking part as given by Eq. (3.9), the bound (3.8) practically applies to both  $D_n^{-1}$  and  $D_{inv,n}^{-1}$ . When using Eq. (3.8) I will disregard the difference between  $\|D_n^{-1}\|$  and  $\|D_{inv,n}^{-1}\|$ .

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<sup>13</sup> The argument may be extended to other theories such as lattice QED with staggered fermions, if a finite but “beyond the Planck scale” lattice cutoff is acceptable.



### C. Recovery of taste symmetry in the continuum limit

Assuming the existence of the continuum limit and the scaling laws (3.8) and (3.9), I will now prove that exact taste symmetry is recovered in this limit for all the coarse-lattice observables of the ordinary staggered theory, provided  $m_r(a_c) > 0$ . The proof makes use of the reweighted theories introduced in Sec. II E, and reveals why it is necessary to avoid an exactly massless lattice theory.

I first add the source term (3.1) to the  $n^{\text{th}}$  reweighted theory (cf. Eq. (2.22a)):

$$Z_{inv,n}(a_c; J) = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] \mathbf{B}_n(1; \mathcal{U}, \{\mathcal{V}^{(k)}\}) \det(D_{inv,n} + J \cdot \mathcal{S}^{(n)}) \quad (3.10)$$

I also introduce a family of partition functions<sup>14</sup> (with sources) in which  $D_n$  is replaced by  $D_{inv,n} + t\Delta_n$ , where  $t$  takes values in the interval  $0 \leq t \leq 1$ . Explicitly,

$$Z_{inter,n}(t, a_c; J) = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] \mathbf{B}_n(1; \mathcal{U}, \{\mathcal{V}^{(k)}\}) \quad (3.11a)$$

$$\begin{aligned} & \times \det(D_n + J \cdot \mathcal{S}^{(n)}) \det \left[ 1 + (t-1)\Delta_n (D_n + J \cdot \mathcal{S}^{(n)})^{-1} \right] \\ & = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] \mathbf{B}_n(1; \mathcal{U}, \{\mathcal{V}^{(k)}\}) \quad (3.11b) \\ & \times \det(D_{inv,n} + J \cdot \mathcal{S}^{(n)}) \det \left[ 1 + t\Delta_n (D_{inv,n} + J \cdot \mathcal{S}^{(n)})^{-1} \right]. \end{aligned}$$

These partition functions interpolate between the reweighted theory for  $t = 0$ , and the staggered theory for  $t = 1$ . Normalized generating functionals are defined in analogy with Eq. (3.5), and the  $n \rightarrow \infty$  limits in analogy with Eq. (3.6).

I will now show that the  $n \rightarrow \infty$  limit does not depend on  $t$ , *viz.*,

$$\mathcal{Z}_\infty(a_c; J) = \mathcal{Z}_{inter,\infty}(t, a_c; J) = \mathcal{Z}_{inv,\infty}(a_c; J). \quad (3.12)$$

The key step is to bound the last factor in Eq. (3.11a) as

$$\exp \text{tr} \log (1 + (t-1)\Delta_n D_n^{-1}) = 1 + O(\epsilon_n^2), \quad (3.13)$$

where

$$\epsilon_n = \left\| D_n^{-1} \right\| \left\| \Delta_n \right\|. \quad (3.14)$$

I have used that  $\Delta_n$  is traceless on the taste index, as well as the geometric series for the staggered propagator  $D_n^{-1} = D_{inv,n}^{-1} - D_{inv,n}^{-1} \Delta_n D_{inv,n}^{-1} + \dots$ . The sources are infinitesimal and do not interfere with any bound valid for  $J = 0$ . The (unnormalized) partition function of any interpolating theory,  $Z_{inter,n}(t, a_c; J)$ , is obtained from the staggered partition function by reweighting with the left-hand side of Eq. (3.13). Because the scaling laws (3.8) and (3.9) hold by assumption on the staggered ensemble, it follows that

$$\epsilon_n \lesssim \frac{a_f}{a_c^2 m_r(a_c)} = \frac{2^{-(n+1)}}{a_c m_r(a_c)}. \quad (3.15)$$

<sup>14</sup> For positivity of the determinants see Sec. V.

We arrive at several important conclusions. First, each term in the Taylor expansion of the logarithm in Eq. (3.13) is bounded by the corresponding power of  $\epsilon_n$ . Second, because the product  $a_c m_r(a_c)$  is held fixed, there will be an  $n_0$  such that, for any  $n \geq n_0$ , one has  $\epsilon_n < 1$  and the Taylor expansion converges. It follows that the change in any (meson) observable over the interval  $0 \leq t \leq 1$  is  $O(a_f^2)$ . Finally, since  $\epsilon_n \rightarrow 0$  for  $n \rightarrow \infty$ , we readily arrive at Eq. (3.12).

The reweighted theories  $\mathcal{Z}_{inv,n}(a_c; J)$  have exact  $U(4)$  taste symmetry by construction, and the same is true for the limiting theory  $\mathcal{Z}_{inv,\infty}(a_c; J)$ . But the limit is independent of  $t$ , and so the staggered generating functional  $\mathcal{Z}_\infty(a_c; J)$  has exact taste symmetry as well; the RG-blocked staggered theory becomes taste-invariant in the limit of infinitely many blocking steps if, in particular, Eq. (3.9) holds. As far as the rate of restoration of taste symmetry is concerned, this is recognized as the familiar result that the discretization errors of staggered fermions are proportional to the (fine) lattice-spacing squared [2, 31, 32, 33, 34].

In the massless staggered theory one has  $m_r(a_c) = 0$ , and the bound (3.15) becomes an empty statement. Therefore it is not possible to infer the recovery of full taste symmetry in the exactly-massless case. This is consistent with the established fact that the continuum and the chiral limits of staggered fermions do not always commute [16, 17, 18, 19].

The result I have established readily generalizes to all the coarse-lattice observables. Given a coarse-lattice operator  $\mathcal{O}^{(n)} = \mathcal{O}^{(n)}(\psi^{(n)}, \bar{\psi}^{(n)}, \mathcal{V}^{(n)})$ , I introduce the notation  $\langle \mathcal{O}_t^{(n)} \rangle_{t'}$  for a “mixed,” unnormalized expectation value where the sea quarks have Dirac operator  $D_{int,n}(t) = D_{inv,n} + t\Delta_n$ , while the valence quarks have Dirac operator  $D_{int,n}(t')$ . To be precise, the Boltzmann weight is given by Eq. (3.11) (with  $J = 0$ ), while the valence propagators are given by  $D_{int,n}^{-1}(t')$ . We then have

$$\langle \mathcal{O}_t^{(n)} \rangle_t = \left\langle \mathcal{O}_t^{(n)} \exp \left[ \text{tr} \log \left( 1 + (t-1)\Delta_n D_n^{-1} \right) \right] \right\rangle_1 \quad (3.16a)$$

$$= \langle \mathcal{O}_t^{(n)} \rangle_1 (1 + O(\epsilon_n^2)) \quad (3.16b)$$

$$= \langle \mathcal{O}_1^{(n)} \rangle_1 (1 + O(\epsilon_n)) . \quad (3.16c)$$

Equation (3.16a) re-expresses an unnormalized observable of  $Z_{inter,n}(t)$  as a correlation function computed on the staggered ensemble with a reweighting factor. Equation (3.16b) follows from the validity of the scaling laws on the staggered ensemble. Last, Eq. (3.16c) replaces any valence propagator  $D_{int,n}^{-1}(t')$  by the staggered propagator  $D_n^{-1}$ . The expansion of  $D_{int,n}^{-1}(t')$  as a power series in  $\Delta_n D_n^{-1}$  has the same convergence properties as that of the logarithm in Eq. (3.13).<sup>15</sup> As a special case,  $\langle 1 \rangle_t = \langle 1 \rangle_1 (1 + O(\epsilon_n^2))$ , and Eq. (3.16) readily generalizes to normalized expectation values. It follows that all the staggered and the reweighted ( $t = 0$ ) observables have the same  $n \rightarrow \infty$  limit, which now establishes the exact taste symmetry of all observables.

Equality of all the observables implies the equality of the “fixed-point” coarse-lattice actions obtained in the limit  $n \rightarrow \infty$ . Comparing once again the staggered and the reweighted

<sup>15</sup> I have allowed for an  $O(\epsilon_n)$  mismatch in the observable, though presumably for any physical quantity of interest one could construct a coarse-lattice observable that would have only  $O(a_f^2)$  discretization errors, in which case the additional mismatch incurred in Eq. (3.16c) is likely to stay  $O(\epsilon_n^2)$  as well.

theories, this means (cf. Eqs. (2.2b) and (2.22b))

$$\begin{aligned} S_\infty(a_c; \mathcal{V}, \psi, \bar{\psi}) &\equiv \lim_{n \rightarrow \infty} S_n(a_c; \mathcal{V}, \psi, \bar{\psi}) \\ &= \lim_{n \rightarrow \infty} S_{inv,n}(a_c; \mathcal{V}, \psi, \bar{\psi}) \equiv S_{inv,\infty}(a_c; \mathcal{V}, \psi, \bar{\psi}). \end{aligned} \quad (3.17)$$

In analogy with Eq. (3.7), I have dropped the blocking-step label attached to the coarse-lattice fields, and traded it with the coarse-lattice spacing. Adding in the source term (3.1) then gives

$$\mathcal{Z}_\infty(a_c; J) = \int \mathcal{D}\mathcal{V} \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ -S_\infty(a_c; \mathcal{V}, \psi, \bar{\psi}) - \bar{\psi} J \cdot \mathcal{S}(a_c; \mathcal{V}) \psi \right]. \quad (3.18)$$

By Eq. (3.17), the action  $S_\infty(a_c; \mathcal{V}, \psi, \bar{\psi})$  of the limiting RG-blocked staggered theory has acquired exact taste- $U(4)$  invariance. Equation (3.18) can be used to derive taste- $SU(4)$  Ward-Takahashi identities that will be exactly satisfied in the limiting theory. The corresponding result in the fourth-root theory (see discussion below Eq. (5.15)) will put on a firm basis the observation made in Refs. [11, 19] that no paradoxes can be derived from the extended taste symmetry of the continuum-limit fourth-root theory.

#### IV. REWEIGHTED THEORIES AT WEAK COUPLING

In the previous section I have shown that, for large enough blocking level  $n$ , the reweighted theory can be reached from the local staggered theory by means of a convergent expansion. The converse is also true: Setting  $t = 1$  in Eq. (3.11b), we can reconstruct the staggered theory from the reweighted theory. The convergence of the ( $t$ )-expansion in Eq. (3.11b) is controlled by  $\epsilon_{inv,n}$ , where

$$\epsilon_{inv,n} = \left\| D_{inv,n}^{-1} \right\|_{inv} \left\| \Delta_n \right\|_{inv}. \quad (4.1)$$

The notation  $\| \cdot \|_{inv}$  means that the norms are now to be evaluated on the *reweighted* ensemble. In view of the established scaling of  $\Delta_n$  on the staggered ensemble, Eq. (3.9), the difference between any staggered-ensemble expectation value and the corresponding reweighted-ensemble expectation value must be very small. Indeed we must have  $\epsilon_{inv,n} \approx \epsilon_n$ , up to corrections which are of higher order (in either of them).

The ability to go back and forth between the staggered and reweighted theories implies that the reweighted theory associated with the local staggered theory must have the following key properties:

- A suitable notion of renormalizability;
- Locality on the coarse-lattice scale;
- Validity of the scaling laws of Sec. IIIB, including in particular the scaling of  $\Delta_n$  as in Eq. (3.9), on the reweighted ensemble.

In this section I explore direct evidence for these key physical properties. The main output is that, step by step, every argument about the (four taste) reweighted theory derived from the local staggered theory generalizes straightforwardly to the (one taste) reweighted version of the fourth-root theory. In Sec. V, this will allow me to establish the validity of

the continuum limit of the fourth-root theory. The  $n \rightarrow \infty$  limit of the blocked fourth-root theory will be reached via the corresponding limit of the sequence of reweighted theories. Since the reweighted theories are all local on the coarse-lattice scale, the same will be true for the (common) limiting theory.

Power-counting renormalizability of the ordinary and fourth-root staggered theories, alongside with the derived reweighted theories, is discussed in Sec. IV A. Locality of the reweighted theories is addressed in Sec. IV B. The scaling laws are discussed in Sec. IV C, relegating some further issues to App. H. I summarize the emerging physical picture in Sec. IV D.

In the rest of this paper I will assume that the coarse-lattice scale has been chosen to satisfy  $a_c \ll \Lambda^{-1}$ . This has the following implications. (1) Because of asymptotic freedom, the running coupling constant  $g_r(a_c)$  is weak at the coarse-lattice scale as well as on all shorter distance scales. (2) One can *define* lattice-regularized QCD to be local if it is local at the coarse-lattice scale. (3) The coarse-lattice observables are rich enough to extract all of the QCD physics.

## A. Renormalizability

I begin with a brief account of what is known about the renormalizability of the ordinary and fourth-root staggered theories (Sec. IV A 1). I then offer a natural definition of renormalizability for reweighted theories, from which it follows that a reweighted theory is automatically renormalizable if the underlying staggered theory is (Sec. IV A 2).

### 1. Staggered theory

Unlike Wilson fermions [35], the task of deriving lattice power-counting theorems and all-orders renormalizability remains to be completed for staggered fermions (for recent progress, see Refs. [13, 36]). Still, it is widely believed that the ordinary staggered theory is renormalizable to all orders. The main evidence comes from a one-loop calculation accompanied by the observation that the staggered-fermion symmetries forbid the generation of any relevant or marginal terms not already present in the staggered action [8]. In particular, the taste-breaking terms remain irrelevant to all orders.

As first noted in Ref. [12], all-orders renormalizability should extend from the ordinary staggered theory to the fourth-root theory. The argument relies on the familiar replica trick.<sup>16</sup> Consider an ordinary staggered theory with  $n_r$  copies, or replicas, of equal-mass staggered fields. At this stage,  $n_r$  is a positive integer. At each order in perturbation theory, the counter-terms needed to renormalize the lattice theory will be polynomials in  $n_r$ , because  $n_r$  only enters as an overall multiplicative factor attached to every closed fermion loop. Next, we consider the analytic continuation in  $n_r$  to arbitrary real values, which corresponds to raising the staggered determinant to a (possibly fractional) power  $n_r$ . This continuation is unique, because the (polynomial!)  $n_r$ -dependence of the diagrammatic expansion at each order, including the counter-terms, is already known. Thus, the counter-terms derived for

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<sup>16</sup> The relevance of the replica trick [37] for the low-energy pion sector of the fourth-root theory has been discussed in Refs. [18, 27, 38].

integer  $n_r$  will be just enough to renormalize the fractional-power theory for *any* value of  $n_r$ . (While this captures the essence of the argument, it amounts to an over-simplification. For a more thorough discussion, see Ref. [11].)

Thus, while the fourth-root theory is non-local [15], renormalizability is *not* lost. Retaining renormalizability turns out to be the absolutely essential starting point from which everything else follows.

## 2. Reweighted theory

Renormalizability of a lattice theory means that, by adjusting the bare parameters, the correlation functions of the renormalized fields have a finite limit when the lattice spacing goes to zero and momenta in physical units are kept fixed, to any order in perturbation theory. In the present RG-blocking context I will assume that the correlation functions under study are constructed from coarse-lattice fields. The external momenta all belong to the Brillouin zone of the coarse lattice.

While the reweighted theories depend on both the fine- and coarse-lattice scales, I will adopt exactly the same criterion to define when they are renormalizable. The implication is that renormalizability carries over automatically from the staggered theory to the derived, reweighted theories. The reason is that, to leading order (in the fine-lattice spacing), the difference between a given coarse-lattice correlation function in the staggered and in the reweighted theory is equal to the taste-breaking part the staggered correlation function. In Sec. IV C below I will argue that the taste-breaking part of any diagram vanishes in the continuum ( $n \rightarrow \infty$ ) limit, and so the staggered and the reweighted theories assign the same continuum-limit value for every correlation function,<sup>17</sup> to any order in perturbation theory.<sup>18</sup> This prediction applies to both the ordinary and the fourth-root staggered theories, in fact to any real value of the number of replicas  $n_r$ .

In order to avoid unrelated non-perturbative complications, as well as to ensure the existence of a weak-coupling regime, I will further restrict  $n_r$  to positive values where the one-loop beta functions (which depends linearly on  $n_r$ ) remains asymptotically free. The main results of Sec. IV B below are valid under these mild restrictions only. As a preparation for the next stage, let me write down the staggered-fermion RG-blocked partition function (without sources) for a general number of replicas  $n_r$ ,

$$Z_n(t, n_r, a_c) = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] \mathcal{B}_n(n_r; \mathcal{U}, \{\mathcal{V}^{(k)}\}) \det^{n_r} (D_{inv,n} + t\Delta_n), \quad (4.2)$$

in which I have also kept the interpolating parameter  $0 \leq t \leq 1$  of Eq. (3.11). All the partition functions studied in this paper are special cases of  $Z_n(t, n_r, a_c)$ . As claimed above, they are all renormalizable.

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<sup>17</sup> The notion of renormalizability is often assumed to include the requirement that the continuum-limit value of each diagram can be made equal to the value computed using some other regularization method by a finite renormalization [35]. To the extent that this is true for the staggered theory, this will readily generalize to the (sequence of) reweighted theories as well.

<sup>18</sup> Extending this claim to the non-perturbative level is the subject of Sec. V.

## B. Locality of $S_{eff}^k$ and $D_k$

The main result of this subsection is that both the effective action  $S_{eff}^k$  (Eq. (2.13)) and the blocked Dirac operator  $D_k$  (Eq. (2.10)) are local on the  $k^{\text{th}}$  lattice scale. In more detail, integrating out fermionic degrees of freedom at the  $k^{\text{th}}$  blocking step generates a local effective action  $S_{eff}^k$  for the gauge field, and the Dirac operator governing the remaining fermionic degrees of freedom is local too. Because  $\tilde{D}_{inv,n}$  is defined by trace projection,  $\tilde{D}_{inv,n}$  and  $\Delta_n$  are separately local. I will argue that this is true on the ensemble generated by  $Z_n(t, n_r, a_c)$  defined above, for any  $t$  and  $n_r$  (in the indicated ranges). The argument relies on the renormalizability of the lattice theory defined by  $Z_n(t, n_r, a_c)$ , but it does not require that that theory be local in itself.

As noted in Sec. II E (see Eq. (2.26)), a corollary of crucial importance is that the theory defined by  $Z_n(t, n_r, a_c)$  turns out to have a local coarse-lattice action whenever raising of the fermion determinant to the  $n_r^{\text{th}}$  power is an analytic operation. A local coarse-lattice action *defines* when a reweighted or an interpolated theory is local. A local coarse-lattice action is obtained for the ordinary ( $n_r = 1$ ) staggered theory, as well as for any theory derived from it by varying  $t$ ; and for the  $n_r = 1/4$  theory at  $t = 0$ , which is recognized as the ( $n^{\text{th}}$ ) reweighted theory derived from the fourth-root theory, cf. Eq. (2.26).<sup>19</sup>

Let me begin with RG-blocking in a pure Yang-Mills theory. While again no rigorous proofs exist, it is widely accepted (see e.g. Ref. [24]) that the pure Yang-Mills lattice action obtained after  $n$  blocking steps is local. The coarse-lattice action can be approximated by the continuum form

$$S_n(a_c) \approx \frac{1}{g_r^2(a_c)} \int d^4x F_{\mu\nu}^2 + \text{discretization errors.} \quad (4.3)$$

The ensemble of coarse-lattice configurations generated by the Boltzmann weight  $B_n(a_c) = \exp(-S_n(a_c))$  will correspond to the correct running coupling  $g_r(a_c)$ . (See App. C for the generation of blocked-lattice gauge field configurations.)

Next let us consider lattice QCD. After integrating over all fields except for the coarse-lattice gauge field, one arrives at a Boltzmann weight of the general form (again using continuum notation)

$$B_n(a_c) \approx \exp \left[ -\frac{1}{g_r^2(a_c)} \int d^4x F_{\mu\nu}^2 + O(1) + \text{discretization errors} \right]. \quad (4.4)$$

Here “ $O(1)$ ” stands for terms occurring at zeroth (or higher) order in the expansion in powers of  $g_r(a_c)$ , which are also of zeroth order in the expansion in powers of the lattice spacing (*i.e.* terms that survive the continuum limit). The form (4.4) depends on renormalizability, because the ultra-violet divergent part of the fermion determinant that renormalizes the coupling constant has been separated out explicitly. All other effects of the integration over

<sup>19</sup> The same is true at  $t = 0$  for  $n_r = n_s/4$ , where  $n_s$  is a positive integer, interpreted as the number of equal-mass sea quarks. All other (fractional!) values of  $n_r$  fail to yield a local action, even at  $t = 0$ , because there is no local Dirac operator  $\tilde{D}$  such that  $\det^{1/n_r}(\tilde{D}) = \det(D_{inv,n})$ . Notice that, in Sec. III, it was not necessary to use the locality of the reweighted theories because the ordinary staggered theory is by itself local.

the fermion fields are contained in the last two terms in Eq. (4.4). These terms obviously include the effects of virtual quark loops, and they are not local. For the considerations in this subsection, the only thing that matters is that the renormalized Yang-Mills action inside of the Boltzmann weight is parametrically larger by  $1/g_r^2(a_c)$ . Equation (4.4) applies to the partition function  $Z_n(t, n_r, a_c)$  as well (after all but the integration over  $\mathcal{V}^{(n)}$  has been done), because this partition function too defines a renormalizable theory.

With Eq. (4.4) in hand, we are ready to discuss the locality properties of  $S_{eff}^k$  and  $D_k$ . Let me summarize the relevant discussion from Sec. II (in particular around Eq. (2.19)), but now in terms of the hermitian [15] operator

$$H_k = [\gamma_5 \otimes \xi_5] G_k^{-1}. \quad (4.5)$$

First, functional differentiation of  $S_{eff}^k$  with respect to the (original or blocked) link variables generates expressions that depend on both  $H_k$  and  $H_k^{-1}$ . Locality of  $S_{eff}^k$  then follows provided that  $H_k$  and  $H_k^{-1}$  are both local on the  $k^{\text{th}}$  lattice scale. The locality of  $D_k$ ,  $H_k$  and  $H_k^{-1}$  is established iteratively using Eqs. (2.11b) and (2.19). (For the  $k = 0$  step, see Ref. [15].) The only non-trivial step is to demonstrate the short-range nature of  $H_k^{-1}$ . In the free theory, this follows because  $H_k$  has an  $O(\alpha_k)$  gap.

In the interacting theory one has to replace the notion of a spectral gap by the notion of a mobility edge. The properties established in the free theory will carry over to any smooth gauge field, and, more technically, to any order in lattice perturbation theory. This leaves open the following question. In the presence of very rough, lattice-size structures in the gauge field, or “dislocations,” could  $H_k$  develop much smaller eigenvalues, which in turn would spoil the short-range character of its inverse? I will now argue that the answer is negative.

Before coming to the main argument I should note that it is logically possible that all eigenvalues  $\lambda_i^{(k)}$  of  $H_k$  may always satisfy  $|\lambda_i^{(k)}| \geq \lambda_{min}^{(k)} > 0$ , for some  $\lambda_{min}^{(k)} = O(\alpha_k)$ . In other words,  $H_k$  might have an  $O(\alpha_k)$  gap for *all* gauge fields. If this is true, we are done. Because it is unknown if this is true, I will disregard this possibility.

According to the theory of disordered systems (see Ref. [39, 40] and references therein), the right question becomes what is the *mobility edge* of  $H_k$ . In general, the spectrum of  $H_k$  will consist of localized eigenstates (at the scale  $a_k$ ) with eigenvalues  $0 \leq |\lambda_i^{(k)}| \leq \lambda_c^{(k)}$ . Above the critical value  $\lambda_c^{(k)}$  the eigenstates become extended. By definition,  $\lambda_c^{(k)} \geq 0$  is the mobility edge of  $H_k$ . The value of  $\lambda_c^{(k)}$  is a property of the ensemble.

I will argue that the mobility edge of  $H_k$  is  $O(\alpha_k)$  in the weak-coupling regime. On general grounds, any localized eigenmodes lying below the mobility edge, be their eigenvalues as small as they may, will not spoil the short-range character of the inverse  $H_k^{-1}$  [39, 41]. The decay length of  $H_k^{-1}$  is thus  $O(a_k)$  as required.

At scales where lattice QCD is weakly coupled, the physics that goes into the mobility edge is simple. Because it may be unfamiliar, and since  $H_k$  itself has not been studied numerically yet, let me digress to describe results obtained in the study of the mobility edge of the hermitian Wilson operator  $H_W$ . In Ref. [40] the mobility edge of  $H_W$  was determined for the super-critical bare mass  $am_0 = -1.5$ , and for a range of values of  $\beta = 6/g_0^2$  on quenched ensembles. This example is relevant for the following reason. First, for the chosen parameters,  $H_W$  and  $H_k$  both have  $O(a^{-1})$  gaps in the free theory, where  $a$  is the relevant lattice spacing. Moreover, at any super-critical bare mass the spectrum of  $H_W$  can, and does, reach zero. Therefore, the analogy will become relevant in case that future numerical work

will demonstrate the existence of low-lying eigenvalues of  $H_k$ . According to the argument below, the corresponding eigenmodes will necessarily be localized.

For the case at hand, the most interesting finding of Ref. [40] was that the mobility edge was very close to the free-theory gap for several different gauge actions, even at the not-very-large cutoff scale  $a^{-1} \sim 2$  GeV. At stronger coupling (lower cutoffs) the mobility edge did go down, eventually reaching zero when the Aoki phase was entered.

Returning to the RG-blocking context, we set the coarse-lattice spacing such that  $g_r(a_c)$  is as weak as we like. Therefore we are interested in values of the mobility edge at weaker couplings than any of those already studied. The results of Ref. [40] suggest that, on a very weakly-coupled pure Yang-Mills ensemble, the mobility edge of any operator, including the super-critical  $H_W$  and  $H_k$ , will be very close to the free-theory gap; for  $\beta \rightarrow \infty$  the mobility edge will continuously approach the free-theory gap.

It remains to consider the inclusion of a fermion determinant raised to some positive (but not necessarily integer!) power at weak coupling, as described by Eq. (4.4). This should have little effect on mobility edges which are already  $O(a^{-1})$ . A different power of a fermion determinant does change the beta-function and the running of the gauge coupling, so we should be a bit careful in what we are comparing. Consider first the operator  $H_n$  of the last blocking step. We may compare the value of its mobility edge on a pure Yang-Mills ensemble to the corresponding value on a dynamical-fermion ensemble that has the same coarse-lattice running coupling  $g_r(a_c)$ . Once the coarse-lattice couplings are equal, the remaining contribution of the fermion determinant in Eq. (4.4) is parametrically smaller by a factor of  $g_r^2(a_c)$  compared to the Yang-Mills action which is the leading term. Changes to the spectrum near a mobility edge which is already  $O(a_c^{-1})$  and, therefore, any further deviations of the mobility edge itself from the free-theory gap, are expected to be very small.

We actually need to know something about the mobility edges of  $H_k$  for all  $k \leq n$ . For each  $k$ , we may compare the blocked staggered ensemble to a new Yang-Mills ensemble chosen such that the running coupling  $g_r(a_k)$  at the  $k^{\text{th}}$  lattice scale is the same in the two theories. Again a similar conclusion will follow.

What can, and will, be significantly affected by the inclusion of fermion determinants is the small-eigenvalue localized spectrum (if there were any near-zero eigenvalues to begin with; see for example Ref. [39]). Since the Boltzmann weight contains  $\det(G_k^{-1}) = \det(H_k)$  raised to a positive power, the transition from the pure Yang-Mills ensemble to dynamical staggered-fermion ensembles will lead to fewer near-zero eigenvalues of  $H_k$ , for all  $k$ .

Let me summarize the anticipated physical situation. In the free theory,  $H_k$  has an  $O(\alpha_k)$  gap. On weakly-coupled pure Yang-Mills ensembles, the mobility edge of  $H_k$  is expected to be very close to the free-theory gap and, thus,  $O(\alpha_k)$  by itself. Now, starting at  $n_r = 0$ , let us gradually increase the power of the staggered-fermion determinant in Eq. (4.2), while maintaining a fixed renormalized coupling  $g_r(a_c)$  at the coarse-lattice scale. Any near-zero eigenvalues of  $H_k$  will be gradually suppressed, but otherwise nothing much should change in the spectrum of  $H_k$ , for all  $k$ . In particular, any further change in the mobility edge will be even smaller than the, by itself small, deviation from the free-theory gap on the pure Yang-Mills ensemble.<sup>20</sup> As a result, the mobility edge of  $H_k$  will remain  $O(\alpha_k)$ , and the decay length of  $H_k^{-1}$  will remain  $O(a_k)$ . As explained above, this implies the locality of  $D_k$ ,  $H_k$ ,  $H_k^{-1}$ , and  $S_{eff}^k$ , for all  $k \leq n$ .

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<sup>20</sup> I use asymptotic freedom to bound  $g_r(a_k)$  by  $g_r(a_c)$ . This restricts my conclusions to the range of  $n_r$ -values where the one-loop beta-function is negative. See also footnote 13.



### C. Scaling of $\Delta_n$

In Sec. IV A I have explained why the partition function (4.2) defines a renormalizable theory for any  $n_r$  and  $t$ . For generic values of these parameters, this theory is non-local. Nevertheless, thanks to its power-counting renormalizability we may study the scaling of any local operator constructed within such a theory. This includes in particular the operators listed at the end of the previous subsection, whose locality I have just established by non-perturbative considerations.

I will be mostly interested in the scaling of  $\Delta_n$ , the taste-breaking part of the blocked Dirac operator, in the staggered ( $t = 1$ ) and in the reweighted ( $t = 0$ ) theory. In this subsection I argue that Eq. (3.9) correctly describes the leading power-law scaling of  $\Delta_n$ . I furthermore find that the logarithmic corrections to the scaling of  $\Delta_n$  depend on  $n_r$  but not on  $t$ . The argument is heuristic, and will have to be confirmed by future calculations.

Setting up perturbation theory is in principle straightforward. In reality, perturbation theory for a reweighted theory is unfamiliar, and technically rather different from ordinary staggered perturbation theory. It can be gradually built up in several steps:

- Step 1.** Ordinary staggered perturbation theory;
- Step 2'.** Staggered perturbation theory with the fermions in a taste basis obtained via a unitary change of variables [6, 32];
- Step 2.** Staggered perturbation theory with the fermions in a taste basis obtained via a gaussian smearing RG-like step [15];
- Step 3.** Multi-gauge-field perturbation theory for the blocked staggered theories of Eqs. (2.8), (2.20) or (2.24);
- Step 4.** Multi-gauge-field perturbation theory for a reweighted theory (or, more generally, for  $Z_n(t, n_r, a_c)$ ).

As I will explain, computing the scaling of  $\Delta_n$  within the fully developed perturbative setup of Step 4 can be reduced, via Steps 3 and 2 (skipping Step 2'), to a calculation in ordinary staggered perturbation theory (Step 1). Focusing on the fourth-root theory, the outcome is that in spite of the lack of a local action, we nevertheless have at our disposal a local operator  $\Delta_n$  that, on the one hand, accounts for all the taste violations in blocked observables, and, on the other hand, is controlled by staggered perturbation theory.

Ordinary staggered perturbation theory is a well-developed technique and there is no need to discuss it here (see e.g. Ref. [11] and references therein). I now discuss all the other steps listed above.

In taste-basis perturbation theory the fermion momentum ranges over a reduced Brillouin zone, and the sixteen sites of each  $2^4$  hypercube are accounted for by the Dirac and the taste degrees of freedom. A taste-basis perturbative expansion is usually not used for two reasons. The unitary transformation to a taste basis is not unique, and complicates the form of many symmetries [6, 32] (see Apps. A, D and E for more details). Also, in a taste-basis diagrammatic expansion, taste violations occur in both the propagator and the vertices, as can immediately be seen by inspection of the free taste-basis Dirac operator (see App. H for further discussion). This is to be contrasted with the usual staggered perturbation theory, where the momentum-space propagator is taste-symmetric and taste violations reside in the vertices only [8, 11].

Next, consider the staggered theory obtained by performing the special  $k = 0$  “blocking” step introduced in Sec. II A for the fermions *only*. No blocking is applied to the gauge field  $\mathcal{U}$ . The resulting Dirac operator is  $D_0$  of Eq. (2.10a). Following the notation of Ref. [15] where its explicit form was derived, in this subsection I denote it as  $D_{taste} = D_{taste}(\alpha_0, m)$ . In the free theory one can write  $D_{taste} = i(\mathcal{A} + \mathcal{B}) + \mathcal{M}$ , where  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{M}$  are all hermitian, and correspond to the  $Dirac \otimes taste$  structures  $\gamma_\mu \otimes \mathbf{1}$ ,  $\gamma_5 \otimes i\xi_\mu \xi_5$ , and  $\mathbf{1} \otimes \mathbf{1}$  respectively. As usual,  $D_{taste}(\alpha_0, m)$  satisfies a GW relation in the massless limit. A key feature of  $D_{taste}(\alpha_0, m)$  is that, provided  $\alpha_0 < \infty$ , dropping the taste-breaking part  $\mathcal{B}$  does *not* introduce any new doublers into the theory [15]. The gaussian-smearing transformation reduces to the previous unitary transformation in the limit  $\alpha_0 \rightarrow \infty$ , where  $\mathcal{M}(\infty, m) = m$ . Here, as in Sec. II, I assume that  $\alpha_0 = O(a_f^{-1})$ .

With its extra technical hassles, staggered perturbation theory with the Dirac operator  $D_{taste}$  is, clearly, highly relevant to the present blocking framework. For the purpose of the discussion below, I only need to draw attention to one fact. By using the general procedure of Ref. [42] one can prove that  $D_{taste}$  retains all the staggered symmetries, albeit in a complicated form.<sup>21</sup> These symmetries forbid the appearance of taste-breaking relevant or marginal terms through loop corrections [8]. The taste-violating part of the fermion self-energy, denoted  $\Gamma_{t.v.}$ , is therefore  $O(p^2)$  in lattice units. But since taste violations occur now in both vertices and propagators, this result necessarily represent a delicate cancellation. Schematically:  $\Gamma_{t.v.}^v - \Gamma_{t.v.}^l = O(p^2)$  where both  $\Gamma_{t.v.}^v$  and  $\Gamma_{t.v.}^l$  are  $O(1)$ , and where  $\Gamma_{t.v.}^v$  accounts for the contribution of diagrams with at least one taste-breaking vertex, while  $\Gamma_{t.v.}^l$  accounts for the remaining contributions, in which taste-breaking arises from the fermion lines only.

The thing to notice is that this delicate cancellation would be hampered had we truncated the free propagator  $(-i(\mathcal{A} + \mathcal{B}) + \mathcal{M})/(\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{M}^2)$  to the “linearized” form  $(-i(\mathcal{A} + \mathcal{B}) + \mathcal{M})/(\mathcal{A}^2 + \mathcal{M}^2)$  obtained keeping only the first taste-breaking term in the geometric-series expansion of the free propagator about the taste-symmetric  $(-i\mathcal{A} + \mathcal{M})/(\mathcal{A}^2 + \mathcal{M}^2)$ . This truncation is not entirely foolish, because it does not introduce doublers into the theory. Yet, it mutilates shift symmetry (see App. H for a related discussion). Dynamically, the reason is that, for lattice-scale momenta that contribute to the loop integrals, the taste-breaking part of the propagator is not small relative to the taste symmetric part. Therefore, truncating the propagator will give rise to large, imbalanced changes in  $\Gamma_{t.v.}^v$  and  $\Gamma_{t.v.}^l$ .

I will now argue that the main change brought about by (many!) iterations of the RG transformation is that the taste-breaking term  $\Delta_n$  becomes *uniformly* small, for all the coarse-lattice momenta. As a result, for large  $n$  the scaling of taste-breaking effects will be controlled by diagrams with a *single* insertion of  $\Delta_n$ .

Before we do any RG-blocked diagrammatic calculation, we must first set up the appropriate perturbative expansion. Usually lattice perturbation theory is based on the expansion of the link variables as  $U_{\mu,x} = \exp(igaA_{\mu,x})$ . When the definition of the lattice theory makes use of the representation (2.8), a similar expansion will have to be applied to the entire tower of gauge fields  $\mathcal{U}, \mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \dots, \mathcal{V}^{(n)}$  simultaneously. With this, one can in principle

<sup>21</sup> The underlying reason is that, on top of the previously applied unitary transformation, gaussian smearing changes the propagator by a contact term only. The long-distance propagator is unchanged, and therefore all the symmetry constraints on long-distance correlation functions must hold, as can be shown using the pull-back mapping (Sec. II C and App. D). The modified, Ginsparg-Wilson-Lüscher (GWL) chiral symmetry [43] associated with the GW relation is a special case of the general construction of Ref. [42].

set up the perturbative expansion for every theory that can be cast in the form of Eq. (4.2), because the closed-form expressions for  $D_k$ ,  $\tilde{D}_k$ ,  $\Delta_k$  and  $S_{eff}^k$  as functionals of all the gauge fields are known.

The next stage is to consider the multi-gauge-field perturbative expansion of the RG-blocked staggered theory ( $t = 1$  in Eq. (4.2)). This perturbation theory will reproduce all the scaling laws derived using ordinary staggered perturbation theory. The reason is simply that, as already noted in Sec. IIIB, one can first integrate out  $\mathcal{U}, \mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \dots, \mathcal{V}^{(n-1)}$ . At this point, one has effectively recovered the diagrammatic expansion derived from the action  $S_n$  (Eq. (2.2b)) of the  $n^{\text{th}}$  level blocked theory. The scaling of the parameters of  $S_n$ , in turn, must agree with the predictions of ordinary lattice perturbation theory.<sup>22</sup> While this description is, strictly speaking, applicable for integer values of  $n_r$ , all other values can be reached via the replica trick, cf. Sec. IV A.<sup>23</sup>

The last step is to show that  $\Delta_n$  must scale in the same way in the (blocked) staggered theory and in the corresponding reweighted theory.<sup>24</sup> This result is established by isolating the diagrams that determine the scaling of  $\Delta_n$  in each theory, and showing that they amount to exactly the same set of diagrams.

In the blocked staggered theory,  $\Delta_n$  gives rise to taste breaking effects either through vertices or through the expansion of the free blocked propagator as  $D_n^{-1} = D_{inv,n}^{-1} - D_{inv,n}^{-1} \Delta_n D_{inv,n}^{-1} + \dots$ . Next comes the main observation. The momentum that flows through any fermion line is, in all cases, a coarse lattice momentum  $p \lesssim 1/a_c$ . After sufficiently many blocking steps, any coarse-lattice momentum will be very small in fine-lattice units:  $pa_f \sim a_f/a_c \ll 1$ . (This is true whether the fermion line forms a closed loop or connects to an external leg.) In contrast,  $\Delta_n$  embodies taste-breaking effects coming from all higher momentum scales up to the fine-lattice cutoff  $1/a_f$ . This means that any mechanism needed to ensure the smallness of all taste breaking effects on the coarse lattice, such as cancellations based on symmetries, must be built into the functional form of  $\Delta_n = \Delta_n(\mathcal{U}, \mathcal{V}^{(0)}, \dots, \mathcal{V}^{(n)})$  itself. Said differently, most of the needed cancellations must occur over distance scales much smaller than  $a_c$ . Therefore, they will not occur in an expectation value with multiple insertions of the operator  $\Delta_n$ , unless they already occurred in every expectation value with a single insertion of  $\Delta_n$ .

Within the multi-gauge-field diagrammatic expansion, this translates into the statement that any insertion of  $\Delta_n$  in any diagram must scale as  $a_f/a_c$  in coarse-lattice units. Given a diagram of the blocked staggered theory, let us now drop every contribution where the total number of insertions of  $\Delta_n$  (coming from both propagators and vertices) is bigger than one. In the remaining taste-violating diagrams, the fermion propagator is  $D_{inv,n}^{-1}$ , and they contain exactly one insertion of  $\Delta_n$ . These diagrams determine the scaling of  $\Delta_n$  in the blocked staggered theory, and, as argued above, will reproduce the taste-breaking scaling laws of the original staggered theory.

But, clearly, the very same set of diagrams is what determines the scaling of  $\Delta_n$  in the *reweighted* theory! I conclude that, in any reweighted theory,  $\Delta_n$  must scale in the same

<sup>22</sup> Alternatively, we may consider the pull-back of  $\Delta_n$  to the original fine-lattice staggered theory, where we may again study its scaling as a function of  $n$ .

<sup>23</sup> As long as we stay within the confines of perturbation theory, this procedure gives meaning to the blocked action  $S_n$  for any real value of  $n_r$ . Conceptually, this is similar to the way the diagrammatic expansion gives meaning to the dimensionally-continued action in dimensional regularization.

<sup>24</sup> The argument can be generalized to  $t \neq 0$ , with the same conclusion.

way as in the original staggered theory. While this will not be needed, the diagrammatic correspondence is clearly tight enough to encompass the logarithmic corrections as well. Thus, the logarithmic corrections depend on  $n_r$ , but not on  $t$ .

More generally, the scaling behavior of any local operator must be the same in the staggered and in the reweighted theory, simply because the scaling will be insensitive to dropping  $\Delta_n$  from the blocked Dirac operator. (The special case considered above amounts to taking the local operator to be  $\Delta_n$  itself.) This implies that the physical observables of the reweighted theory should have scaling violations proportional to  $a_f^2$ , as predicted by (perturbation theory for) the original staggered theory. The  $n \rightarrow \infty$  limit therefore yields a “perfect action” theory [24]. (See, however, App. D for a discussion of related technical issues.)

In summary, we learn two important lessons. After many blocking steps,  $\Delta_n$  will be small on any staggered or reweighted ensemble. We may thus compute its scaling behavior on either ensemble by appealing to perturbation theory for the multi-gauge-field representation of the blocked staggered theory ( $t = 1$ ). Also, as long as we allow for coarse-lattice observables only, this calculation further reduces to a conventional scaling calculation in staggered perturbation theory, augmented by the replica trick (for non-integer  $n_r$ ).

In particular, I find that the power-law scaling (3.9) is valid in the reweighted theory derived from the fourth-root theory. Notice that I have assumed that  $\Delta_n$  scales like a dimension-five (and not like a dimension-six) operator. But, as explained in Sec. III C, thanks to taste-tracelessness of  $\Delta_n$  this assumption is consistent with  $O(a_f^2)$  scaling of the taste-violating effects in all the physical observables. In Sec. V, the scaling of  $\Delta_n$  will be used to establish the validity of the continuum limit of the fourth-root theory.

In this subsection I gave only a very minimal discussion of the multi-gauge-field diagrammatic expansion. In App. H, I illustrate some further aspects of this expansion by considering a few examples of terms which are expected to occur in  $\Delta_n$ .

## D. Summary and future work

We are almost done. In the next section, the reweighted theories will be used to establish the validity of the fourth-root theory in the continuum limit. This conclusion is a straightforward corollary of the emerging physical picture of the reweighted theories. I therefore pause to summarize what has been learned.

All the reweighted theories introduced in Sec. II E share the following key features: (1) renormalizability, (2) locality, and (3) the same scaling laws as the underlying staggered theory. At the starting point is the all-essential observation that the fourth-root theory is renormalizable if the ordinary staggered theory is. From this point on, basically the same reasoning was applied in both the ordinary and fourth-root cases. In fact, for the most part the arguments generalize to any real number of replicas  $n_r$  within the range specified above Eq. (4.2).

As briefly discussed in Sec. IV A 1, renormalizability of both the ordinary and fourth-root staggered theories is not as solidly established as in other cases (notably Wilson fermions). But there is no real reason to doubt it either. For a recent, more thorough discussion, see Ref. [11]. As explained in Sec. IV A 2, the reweighted theories “inherit” their renormalizability from the underlying staggered theory, in a rather trivial way.

Locality of the reweighted theories at the coarse-lattice scale rests on the locality of the effective action  $S_{eff}^k$  and the blocked Dirac operator  $D_k$ , on the relevant ensemble. Those locality properties, in turn, are set by the range of  $H_k^{-1}$ , where the hermitian operator  $H_k$

(Eq. (4.5)) accounts for the short-distance fermion modes integrated out at the  $k^{\text{th}}$  step. I have argued in Sec. IV B that what is needed is that the *mobility edge* of  $H_k$  be  $O(1)$  in units of the  $k^{\text{th}}$  lattice scale. I have drawn an analogy to a recent application of the theory of localization to lattice QCD, specifically, to a study of the mobility edge of the super-critical Wilson operator [39, 40]. I concluded that, thanks to the existence of a weak-coupling regime (which in turn is a consequence of renormalizability), both the mobility edge of  $H_k$  and the range of  $H_k^{-1}$  will be  $O(1)$  in  $k^{\text{th}}$  lattice units, as required. Obviously, it will be necessary to confirm the claims by numerical investigations of  $H_k$  itself. The first non-trivial instance is provided by the  $k = 1$  blocking step.<sup>25</sup>

Our knowledge about the ordinary staggered theory strengthens the claims I have made. In the ordinary staggered theory, based on standard RG considerations one assumes that the blocked action  $S_n$  (Eq. (2.2b)) will be local on the coarse-lattice scale. For this to be true, the locality properties of  $S_{eff}^k$  and  $D_k$  must be as claimed. But my reasoning in Sec. IV B did not discriminate between the ordinary and the fourth-root ensembles. This lends higher credibility to the proposed physical picture in the fourth-root case as well.

My claims are on stronger footing for  $n_r = 0$  as well: this quenched limit is closer to the actual setup of the work reported in Ref. [40]. The fourth-root value  $n_r = 1/4$  may thus be reached by interpolation, starting either from  $n_r = 0$  or  $n_r = 1$ . Once again, this supports the claims made in the fourth-root case.

I now turn to the scaling of the taste-breaking effects represented by  $\Delta_n$ . The basic difficulty is simply that, in the fourth-root theory, there is no local fermion action. Thus, it is unclear if the taste violations that reside in the fermion sector are amenable to a scaling analysis.

First, the fourth-root theory is renormalizable. Therefore, even though the theory is non-local, we have a power counting and we can study the scaling of any local operator.<sup>26</sup> Specifically, I have shown that a scaling analysis in the fermion sector is made possible thanks to the multi-gauge-field representation introduced in Sec. II. This gives us access to the operator  $\Delta_n$  that accounts for all the taste-symmetry violations in blocked observables. According to the discussion of Sec. IV B,  $\Delta_n$  is a *local* operator; therefore its scaling can be computed using the appropriate (multi-gauge-field) perturbative expansion. Finally, I have argued that the needed scaling calculation ultimately reduces to a calculation in ordinary staggered perturbation theory augmented by the replica trick, and that  $\Delta_n$  indeed scales as an irrelevant operator should.

My arguments in Sec. IV C were heuristic, and it is clearly necessary to confirm them by performing the appropriate perturbative calculations. The actual scaling of  $\Delta_n$  can also be investigated numerically, at least on the fourth-root ensembles provided by MILC [44]. A first study was performed last year [30]. Most of the arguments of this section rely on being in a (sufficiently) weak-coupling regime, and it is important to understand how close are we to this region in practice. Numerically reweighting is clearly a challenge, which, if successfully tackled, could further strengthen confidence in the entire framework. Another challenging project is to perform an accurate comparison of the predictions of the various perturbative expansions to numerical results obtained e.g. by measuring Wilson loops [36] or by adapting the Schrödinger-functional technique [45, 46, 47, 48].

<sup>25</sup> Because of special features of the  $k = 0$  step, the operator  $H_0$  is guaranteed to have a gap in the interacting theory too. The same is not true for  $k \geq 1$ .

<sup>26</sup> See Ref. [11] for similar examples taken from Condensed Matter physics.

## V. CONTINUUM LIMIT OF THE FOURTH-ROOT THEORY

Assuming the properties of the reweighted theories discussed in Sec. IV, in this section I prove the validity of the fourth-root theory in the continuum limit. As explained earlier, when the blocking level  $n$  is high enough, one can either reach the reweighted theory from the staggered theory via a convergent expansion, or work the other way around. I find it appealing to reconstruct the fourth-root theory from the reweighted theory, because the latter is local, and is already expected to be in the correct universality class. The argument, that otherwise follows the same logic as in Sec. III, is given in Sec. V A. I add several comments on the scaling analysis in Sec. V B.

### A. Recovery of locality in the continuum limit

In the fourth-root theory I will make use of the scaling laws (compare Eqs. (3.8) and (3.9))

$$\|\Delta_n\|_{inv} \lesssim \frac{a_f}{a_c^2}, \quad (5.1)$$

where the subscript “ $inv$ ” refers to the reweighted ensemble, and

$$\|D_{inv,n}^{-1}\|_{inv} \lesssim \frac{1}{m_{inv,n}}, \quad (5.2)$$

where

$$m_{inv,n} = m_r(a_c) + O(a_f/a_c^2). \quad (5.3)$$

The scaling of  $\Delta_n$  was discussed in Sec. IV C. The leading power-law behavior of  $\Delta_n$  is robust. It is unchanged by taking the fourth root, and it is also independent of reweighting. Turning to Eq. (5.3), the origin of the rightmost term is simply that the transition from  $D_n$  to  $D_{inv,n}$  amounts to dropping  $\Delta_n$ . The latter is  $O(a_f/a_c^2)$  which, therefore, could entail similar changes in the eigenvalues. See App. G for some further comments on the bound (5.3). Similar considerations show that the effective coupling constant of the reweighted theory, denoted  $g_{inv,n}$ , satisfies

$$g_{inv,n}^2 = g_r^2(a_c) [1 + O((a_f/a_c)^2)], \quad (5.4)$$

where I have used Eq. (4.4) and the taste-tracelessness of  $\Delta_n$ . This implies that the reweighted theory is in a weak-coupling regime if the coarse-lattice staggered theory is, and vice versa.

I will restrict the present discussion to meson observables of the fourth-root theory. The generalization to all other observables requires additional technical steps which are discussed in Appendix B of Ref. [11]. The physical meson observables are taste singlets. They are probed by restricting the source term of Sec. III A to the form  $\tilde{J} \cdot \mathcal{S}^{(n)}$ , where now  $\mathcal{S}_i^{(n)} = [\tilde{\mathcal{S}}_i^{(n)} \otimes \mathbf{1}]$ . Here  $\tilde{\mathcal{S}}_i^{(n)}$  carries no taste index and, as usual,  $\mathbf{1}$  is the identity matrix in taste space. Switching notation from  $J$  to  $\tilde{J}$  is meant to remind us that the sources now couple to taste singlets only. The blocked fourth-root partition function with these sources is given by

$$Z_n^{root}(a_c; \tilde{J}) = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] B_n\left(\frac{1}{4}; \mathcal{U}, \{\mathcal{V}^{(k)}\}\right) \det^{1/4}\left(D_n + \tilde{J} \cdot \mathcal{S}^{(n)}\right). \quad (5.5)$$

Equation (5.5) means that observables are constructed as follows [27]. Fermion – anti-fermion contractions are done in the same way as in the ordinary staggered theory; then one applies the extra “replica” rule that a factor of  $\frac{1}{4}$  is to be attached to every closed fermion loop occurring in the observable itself (in other words, to every valence staggered-fermion loop). With this replica rule in place, the pull-back mapping defined in Eqs. (2.14) and (B1) remains valid, and the same is true for Eq. (2.15). Thus, the ultra-local nature of the pull-back mapping is preserved, even though the lattice action itself is not local.

I add the same source term to the reweighted theories, which gives rise to

$$Z_{inv,n}^{root}(a_c; \tilde{J}) = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] B_n\left(\frac{1}{4}; \mathcal{U}, \{\mathcal{V}^{(k)}\}\right) \det\left(\tilde{D}_{inv,n} + \tilde{J} \cdot \tilde{\mathcal{S}}^{(n)}\right). \quad (5.6)$$

Here I have used the exact taste invariance of the reweighted theories and the taste-singlet nature of the sources to take the analytic fourth root. In analogy with Sec. III, I also introduce interpolating theories (with the same source), whose partition functions can be expressed as:

$$\begin{aligned} Z_{inter,n}^{root}(t, a_c; \tilde{J}) &= \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] B_n\left(\frac{1}{4}; \mathcal{U}, \{\mathcal{V}^{(k)}\}\right) \\ &\times \det\left(\tilde{D}_{inv,n} + \tilde{J} \cdot \tilde{\mathcal{S}}^{(n)}\right) \det^{1/4}\left[1 + t\Delta_n\left(D_{inv,n} + \tilde{J} \cdot \mathcal{S}^{(n)}\right)^{-1}\right]. \end{aligned} \quad (5.7)$$

Normalized varieties of all partition functions are defined in analogy with Eq. (3.5), e.g.

$$\mathcal{Z}_{inv,n}^{root}(a_c; \tilde{J}) = Z_{inv,n}^{root}(a_c; \tilde{J}) / Z_{inv,n}^{root}(a_c; 0). \quad (5.8)$$

I will assume that the continuum limit of the (sequence of) reweighted theories exists:

$$\mathcal{Z}_{inv,\infty}^{root}(a_c; \tilde{J}) = \lim_{n \rightarrow \infty} \mathcal{Z}_{inv,n}^{root}(a_c; \tilde{J}). \quad (5.9)$$

As usual, this is based on the scaling of the coupling constant itself, which, in turn, is only negligibly affected by reweighting (cf. Eq. (5.4)).

We are now ready to reconstruct the observables of the fourth-root staggered theory from those of the reweighted theory. To this end I use that, on the reweighted ensemble,

$$\exp\left[\frac{1}{4}\text{tr} \log\left(1 + t\Delta_n D_{inv,n}^{-1}\right)\right] = 1 + O\left((t\epsilon_{inv,n})^2\right). \quad (5.10)$$

The definition of  $\epsilon_{inv,n}$  is the same as in Eq. (4.1), except that this is now in the context of the fourth-root theory of course. The similarity between Eqs. (3.13) and (5.10) is clear. It follows that

$$Z_{inter,n}^{root}(t, a_c; \tilde{J}) = Z_{inv,n}^{root}(a_c; \tilde{J}) \left[1 + O\left((t\epsilon_{inv,n})^2\right)\right]. \quad (5.11)$$

Because  $m_r(a_c)$  scales logarithmically while  $\Delta_n$  is suppressed by a power of the fine-lattice cutoff (cf. Eq. (5.1)), it is guaranteed that, for  $n$  above a certain value, we will have  $\epsilon_{inv,n} < 1$  and, with it, convergence of the  $t$ -expansion in Eq. (5.10). Once again,  $\epsilon_{inv,n} \rightarrow 0$  for  $n \rightarrow \infty$ , and the continuum limit is independent of  $t$ ,

$$\mathcal{Z}_{\infty}^{root}(a_c; J) = \mathcal{Z}_{inter,\infty}^{root}(t, a_c; J) = \mathcal{Z}_{inv,\infty}^{root}(a_c; J). \quad (5.12)$$

We now recall that the reweighted theories are local on the coarse-lattice scale, as can be seen from the path integral representation (2.26), and that they belong to the correct universality class. In the limit  $n \rightarrow \infty$  we thus have

$$\mathcal{Z}_{inv,\infty}^{root}(a_c; \tilde{J}) = \int \mathcal{D}\mathcal{V} \mathcal{D}q \mathcal{D}\bar{q} \exp \left[ -S_{inv,\infty}^{root}(a_c; \mathcal{V}, q, \bar{q}) - \bar{q} \tilde{J} \cdot \tilde{\mathcal{S}}(a_c; \mathcal{V}) q \right], \quad (5.13)$$

where the limiting action

$$S_{inv,\infty}^{root}(a_c; \mathcal{V}, q, \bar{q}) = \lim_{n \rightarrow \infty} S_{inv,n}^{root}(a_c; \mathcal{V}, q, \bar{q}), \quad (5.14)$$

is local too. But, by Eq. (5.12),  $\mathcal{Z}_{inv,\infty}^{root}(a_c; \tilde{J})$  accounts for the continuum-limit observables of the (blocked) fourth-root theory as well. This establishes the validity of the continuum limit of the fourth-root theory.<sup>27</sup>

I conclude with two additional observations. My first comment concerns the physical consequences of the continuum-limit taste symmetry of the fourth-root theory. One can lift the restriction on the sources and consider (meson) observables with a general taste structure. The reweighted partition function then takes the form

$$Z_{inv,n}^{root}(a_c; J) = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n \left[ \mathcal{D}\mathcal{V}^{(k)} \right] B_n \left( \frac{1}{4}; \mathcal{U}, \{\mathcal{V}^{(k)}\} \right) \det^{1/4} \left( \tilde{D}_{inv,n} \otimes \mathbf{1} + J \cdot \mathcal{S}^{(n)} \right). \quad (5.15)$$

Taste- $SU(4)$  Ward-Takahashi identities may now be derived by varying  $J$ . These identities will be exact in the reweighted theory for every  $n$ , and will be true up to  $O(\epsilon_{inv,n}^2)$  corrections in the blocked fourth-root theory. In the  $n \rightarrow \infty$  limit, these identities will become exact in the fourth-root theory as well. If, however, we reinstate the restriction to taste-singlet observables, then Eq. (5.15) evidently reduces to Eq. (5.6). This means that no paradoxes can be derived based on the taste symmetry of the continuum-limit fourth-root theory (as claimed in Ref. [49]). The taste non-singlet states leave in an extended, non-unitary Hilbert space; but a unitary, physical sub-space exists. A more practical conclusion concerns deciding when one is allowed to use taste non-singlet operators (such as those in Eq. (D2)), which are often advantageous numerically, instead of taste-singlets ones. For a detailed discussion of these issues, see Refs. [11, 19, 21].

Another observation is that, as I have assumed in Sec. IIE,  $\det(D_{inv,n})$  and  $\det(\tilde{D}_{inv,n})$  will both be strictly positive when  $\epsilon_n, \epsilon_{inv,n} < 1$ . I begin by re-writing the blocked staggered determinant as

$$\det(D_n) = \det(D_{inv,n}) \det \left( 1 + \Delta_n D_{inv,n}^{-1} \right). \quad (5.16)$$

Because  $[\gamma_5 \otimes \xi_5] D_n$  is hermitian [15], and  $D_{inv,n} = \tilde{D}_{inv,n} \otimes \mathbf{1}$  accounts for its taste-invariant part, it follows that  $\gamma_5 \tilde{D}_{inv,n}$  is hermitian too. Therefore,  $\det(D_{inv,n})$  and  $\det(\tilde{D}_{inv,n})$  are real. Moreover, by my assumption, the expansion of the determinant in Eq. (5.16) is convergent (on both the staggered and reweighted ensembles), and the rightmost determinant in Eq. (5.16) is thus strictly positive. Since  $\det(D_n)$  is strictly positive too (see Sec. IIB), it follows that  $\det(D_{inv,n})$  is strictly positive. Next I consider  $\det(\tilde{D}_{inv,n})$ . Because  $\det^4(\tilde{D}_{inv,n}) = \det(D_{inv,n})$ , we know that  $\det(\tilde{D}_{inv,n})$  cannot be zero and cannot flip

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<sup>27</sup> As explained in Sec. IV C,  $S_{inv,\infty}^{root}$  is a “perfect” action.



sign. By considering the limit where the bare mass of the original staggered theory goes to infinity, it follows that  $\det(\tilde{D}_{inv,n})$  is strictly positive.

Finally I should note that it is quite certain that the bounds I have made use of in this section and in Sec. III must represent over-estimations. I return to this point in Sec. VI.

## B. Scaling in the reweighted theories revisited

A key result of this paper is that the fermion sector of the fourth-root theory becomes amenable to a scaling analysis by means of the multi-gauge-field representation of the blocked theory. As explained in Sec. IV C, the scaling analysis in the fourth-root theory can be carried out by reducing it to a calculation in ordinary staggered perturbation theory augmented by the replica trick.

Interestingly, in the reweighted fourth-root theory, the needed scaling laws may be found without making any reference to the replica trick in staggered perturbation theory. According to this alternative route, the calculation of the scaling of  $\Delta_n$  (which is still done as described in Sec. IV C) proceeds by first considering only reweighted theories with  $n_s = 4n_r$  quark species, where  $n_s$  is a multiple of four, and therefore  $n_r$  is integer. This means that the complete calculation, including the part done on the staggered-theory side, involves local theories only. The scaling of  $\Delta_n$  for any other number of quark species  $n_s$  in the reweighted theory can now be found without any further reference to the staggered theory. We simply analytically continue the previous result to the desired value of  $n_s$ . As usual, because the  $n_s$  dependence is known in closed form, the analytic continuation is uniquely determined. Unlike in the staggered theory, however, this analytic continuation only relates local (reweighted) theories to other local (reweighted) theories! In particular, the scaling in the one-taste reweighted theory is inferred from the scaling in reweighted theories where the number of quark species is a multiple of four, without ever having to perform a scaling analysis in the fourth-root theory.

Thus, this line of argument relates the needed scaling properties of the reweighted fourth-root theory to the local, ordinary staggered theory, while passing only through local theories at intermediate steps. The fourth-root theory is then encountered only at the very last stage, where we reconstruct it from the reweighted theory, as was done in Sec. V A.

I comment in passing that, “forgetting” where they came from, the reweighted theories  $Z_{inv,n}$  or  $Z_{inv,n}^{root}$  each constitute a family of local theories defined on a lattice with spacing  $a_c$ , which depend on an additional parameter  $n$ . The role of this parameter is similar to the fifth dimension  $L_5$  of domain-wall fermions: when either  $n$  or  $L_5$  are sent to infinity, a GWL chiral symmetry is recovered.<sup>28</sup> The actual construction of the reweighted theories would amount to a gross “overkill,” if our only aim was to find solutions of the GW relation. The merit of the construction is that the same local operator,  $\Delta_n$ , controls both the violations of the GWL chiral symmetry (that originates from the staggered  $U(1)_\epsilon$  symmetry) in the reweighted theory, and the deviations of the latter from the corresponding staggered theory.

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<sup>28</sup> Of course, in a one-flavor theory, a GW symmetry exists only in the free theory.

## VI. CONCLUSION

Like a journey through a dark wood, when dealing with a difficult problem in quantum field theory one can never be too sure which is the right way, and where danger is lurking. I have concluded that the fourth-root recipe is valid in the continuum limit using plausible assumptions. Plausibility is, at the end of the day, in the eye of the beholder. In this concluding section, I give my personal perspective on what has been gained.

In a way, this paper trades one set of questions for another. But while at the starting point the questions were rather vague, the new questions are focused, technical, and testable. The initial worries basically stem from our lack of experience with non-local theories. A formal expansion of the staggered action suggests that the taste-breaking terms are irrelevant operators, that would naively be expected to vanish in the continuum limit. But it is unclear how to perform a scaling analysis when there is no local fermion action in the first place. Related, one must also translate the (tentative) claim “locality is recovered in the continuum limit” into a well-defined statement.

This paper offers a solution to these problems. By first RG-blocking the (fourth-root) staggered theory and then enforcing exact taste symmetry by reweighting, we obtain local coarse-lattice theories in the desired universality class, which provide a good approximation of the (fourth-root) staggered theory once the number of blocking steps is large enough. That the reweighted and staggered theories are indeed close to each other, follows from a scaling analysis, which, in the fourth-root case, is made possible by the multi-gauge-field representation of the blocked theory introduced in Sec. II. Within this representation, the taste-breaking effects all arise from the taste-breaking part  $\Delta_n$  of the local, blocked Dirac operator  $D_n$ .

The reasoning of this paper has been presented early on in Sec. II E and I now recapitulate it: All-orders renormalizability of the reweighted theories follows from that of the (ordinary and fourth-root) staggered theories (Sec. IV A); making mild use of renormalizability to establish the existence of a weak-coupling regime, a robust non-perturbative consideration shows that the reweighted theories are local (Sec. IV B); the scaling of the local operator  $\Delta_n$ , which embodies all the taste violations in the blocked theory, can be traced back to a calculation in ordinary staggered perturbation theory (augmented by the replica trick in the case of the fourth-root theory), and the result is that  $\Delta_n$  indeed scales as an irrelevant operator (Sec. IV C); the smallness of  $\Delta_n$  on the reweighted ensemble enables the reconstruction of the staggered theory from the reweighted theory by means of a convergent expansion (Sec. V); in the continuum limit, the difference between the (blocked) staggered theory and the reweighted theory, which is already known to be in the correct universality class, vanishes. For the ordinary, local staggered theory this implies that exact taste symmetry has been recovered; for the fourth-root theory, this implies that it has become local. Thus the fourth-root theory provides a valid regularization of QCD.

This conclusion depends on confirming the key properties of the reweighted theories. This amounts to verifying their locality, checking the actual predictions of their perturbation theory, as well as testing these predictions non-perturbatively (by numerical methods). A summary of what each of the above amounts to has been given in Sec. IV D.

A detailed-level comprehensive study of all the properties of the reweighted theories would be a major endeavor. Nevertheless, already now there is good reason to believe that the fourth-root theory is indeed a valid regularization of QCD. This conclusion derives from the comparison to the local four-taste staggered theory. In short, our understanding of the local

staggered theory is on essentially the same footing as with any other local lattice fermion method. If the continuum limit of the four-taste staggered theory is what we think it is, then it is difficult to see how the claimed properties of the reweighted theories derived from it could go wrong. But then, the arguments for the key properties of the reweighted theories apply, basically unchanged, to the four-taste (derived from local staggered) and one-taste (derived from fourth-root staggered) cases, both of which constitute local theories. Thus, it is also difficult to see how any key property of a reweighted theory could go wrong in the one-taste case, if this does not happen in the four-taste case.

By following this line of argument one can in fact avoid any reference to the replica trick – which is the manifestation of the non-locality in staggered perturbation theory. Instead, one first derives the scaling laws for reweighted theories where the number of quark species is a multiple of four (and, thus, the original staggered theory is local). From this, one infers the scaling laws for reweighted theories with any other number of quark species (Sec. VB). Thus, any reference to the non-local fourth-root theory is avoided until the very last step where it is reconstructed from the reweighted theory.

Taste-breaking effects in the spectrum of the staggered Dirac operator are largest at the (fine-lattice) cutoff scale. But the largest taste-breaking effects are *not* a major source of non-locality; in fact they entail basically no non-locality, because RG blocking trades all ultra-violet fermion modes with a local correction to the gauge-field action. This observation is nothing but a (part of the standard) description of how symmetries broken by the lattice regulator are recovered in the continuum limit. A key result is that this feature is not lost by the fourth-root theory.

The remaining non-local effects have been argued to be associated with the dimensionless, small parameter  $a_f\Lambda$  [15, 21, 27, 28]. It follows from the results of this paper that *all* the non-local effects should be controlled by (powers of)  $a_f\Lambda$ . In Sec. III and Sec. V, I have bounded the relative size of taste-breaking effects in long-distance observables, hence also the relative size of non-local effects, by powers of  $a_f/(a_c^2 m_r(a_c))$ . But, because the coarse-lattice spacing  $a_c$  is basically arbitrary (apart from the restriction  $a_c \ll \Lambda^{-1}$ ), this has got to be an over-estimation. In all likelihood, the actual relative size of the taste-breaking and the non-local effects is on the order of  $a_f\Lambda^2/m_{phys}$  (or powers thereof), where  $m_{phys}$  is the renormalized quark mass extract from some low-energy observable. This is based on the anticipation that, on low-energy modes of the staggered Dirac operator, the actual magnitude of taste-splittings among quartets of eigenvalues should scale like  $a_f\Lambda^2$ . For related theoretical discussions, see Refs. [15, 19, 50].

In numerical simulations, the taste-symmetry violations are observed to decrease rapidly as  $a_f$  is decreased, and indeed to be roughly proportional to  $(\alpha_s(a_f) a_f\Lambda)^2$  (with “improved” staggered quarks) [9]. The presence of a fixed physical scale, and not some  $a_c^{-1} \gg \Lambda$ , makes it possible to extrapolate to the continuum limit using present-day computer resources.

This work lends strong support to the physical picture advocated in Refs. [19, 21, 27]: The non-localities of the fourth-root theory can be interpreted in terms of an extended Hilbert space containing states with, in general, non-zero taste charges. The physical subspace consists of the taste-singlet states. The exact taste symmetry, recovered in the continuum limit, relates physical and unphysical states, and its Ward-Takahashi identities play a crucial role in establishing unitarity in the physical subspace.

It is interesting to consider a closely related problem, namely the use of the fourth-root recipe for finite-density simulations (see Ref. [50] and references therein). Here there is a new, three-fold difficulty. First, there are all the general difficulties having to do with a complex

measure, that set in when  $\text{Re}(\mu) \neq 0$ , where  $\mu$  is the chemical potential. Second, when trying to apply the fourth-root recipe one confronts phase ambiguities, and the systematic error they introduce must be kept under control.<sup>29</sup> Last, a non-zero quark mass is no longer an effective infra-red cutoff. The eigenvalues reach the origin in the complex plane for realistic values of the chemical potential. Even in this case, it has been argued in Ref. [50] that everything is *in principle* under control, provided that the continuum limit is taken before the thermodynamical limit. The crucial grouping of eigenvalues into quartets – near the origin in the complex plane and beyond – can still be done when one is close enough to the continuum limit, if the volume (in physical units) is finite. However, the systematic error due to the phase ambiguities is parametrically much larger, and grows with a positive power of the volume. For more details, see Ref. [50].

Returning to zero density, the up and down quark masses used in numerical simulations are larger than their physical values. Extrapolation of numerical results to the physical point requires the appropriate low-energy effective theory. For the development of staggered chiral perturbation theory (S $\chi$ PT) see Refs. [12, 34, 38]. Recently, based on plausible assumptions within the context of the chiral effective theory, it has been argued that S $\chi$ PT augmented by the replica trick is indeed the correct low-energy description of the pion sector of the fourth-root theory [27].

It will be interesting to re-derive S $\chi$ PT with the replica trick directly from the underlying theory, the (RG-blocked) fourth-root theory. The difficulty is that the effective theory depends on the number of replicas  $n_r$  both explicitly, as well as implicitly through the  $n_r$ -dependence of its low-energy constants. Normally, the dependence of low-energy constants on the parameters of the underlying theory is non-perturbative, and is not known. The challenge is to cast the (RG-blocked) underlying theory into a new form where the necessary analytic continuation in the number of fermion species of some type can be done in a closed form. Work on this subject is in progress [51].

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<sup>29</sup> This difficulty never appears when  $\text{Re}(\mu) = 0$ . When the quark mass is strictly positive the staggered determinant is strictly positive too, and the positive, analytic fourth root can always be chosen.

## APPENDIX A: THE FERMION BLOCKING KERNELS

In this appendix I describe the fermion blocking kernels in some more detail. The transition to a taste representation in the special  $k = 0$  step is discussed in App. A 1. The blocking kernels of all subsequent steps are introduced in App. A 2. In App. A 3, I prove the positivity of  $\det(D_n)$  and  $\det(G_n^{-1})$ .

### 1. Taste representation in the interacting theory

In the case of free staggered fermions there is a unitary transformation between the one-component field  $\chi(x)$  and the taste-basis field  $\psi_{\alpha i}^{(0)}(\tilde{x}^{(0)})$ , given explicitly by [5, 6]

$$\psi_{\alpha i}^{(0)}(\tilde{x}^{(0)}) = \sum_{r_\mu=0,1} (\gamma_1^{r_1} \gamma_2^{r_2} \gamma_3^{r_3} \gamma_4^{r_4})_{\alpha i} \chi(2\tilde{x}^{(0)} + r). \quad (\text{A1})$$

For notation see Sec. II A. Writing Eq. (A1) compactly as  $\psi = \Gamma \chi$ , the “conjugate” Grassmann variables are related by  $\bar{\psi} = \bar{\chi} \Gamma^\dagger$ . Recall that the fine-lattice spacing  $a_f$  of the one-component field, and the lattice spacing  $a_0$  of the taste-basis field, are related via  $a_0 = 2a_f$ . Equation (A1) makes use of the embedding of the taste-basis lattice into the fine lattice, and the fact that each fine-lattice site has a unique representation as  $x_\mu = 2\tilde{x}_\mu^{(0)} + r_\mu$ , where  $r_\mu = 0, 1$ .

In a free theory, RG blocking normally works by suppressing modes with a lattice-scale momentum. It is therefore natural to apply the blocking in the taste basis [20], where all the long-distance physics comes from the vicinity of the origin in the Brillouin zone. The one-component formalism would be inconvenient<sup>30</sup> because the long-distance physics comes from all sixteen “corners” of the Brillouin zone.

Unlike the free theory, an equal choice between a one-component basis and a taste basis does not exist in the interacting theory. Lattice QCD with staggered fermions must be defined in the one-component formalism. According to power-counting arguments and explicit one-loop calculations, only this formalism has enough symmetry to ensure the multiplicative renormalization of the staggered-fermion mass term and the recovery of full rotation and taste symmetries in the continuum limit [8, 53]. This state of affairs poses a difficulty for the RG program. The question is how to accommodate all the symmetries of the standard one-component formalism in a taste-basis representation that will, in turn, provide the starting point for the succession of RG-blocking steps.

The unitary transformation from the one-component basis to the taste basis can be promoted to a gauge-covariant one,

$$\begin{aligned} \psi_{\alpha i}^{(0)}(\tilde{x}^{(0)}) &= Q_{\alpha i}^{(0)}(\tilde{x}^{(0)}) \chi \\ &\equiv \sum_{r_\mu=0,1} (\gamma_1^{r_1} \gamma_2^{r_2} \gamma_3^{r_3} \gamma_4^{r_4})_{\alpha i} \mathcal{W}(2\tilde{x}^{(0)}, 2\tilde{x}^{(0)} + r; \mathcal{U}) \chi(2\tilde{x}^{(0)} + r). \end{aligned} \quad (\text{A2})$$

Setting  $x_0 = 2\tilde{x}^{(0)}$  for short, an explicit choice for the parallel transporter is [6]

$$\mathcal{W}(x_0, x_0 + r; \mathcal{U}) = U_{1,x_0}^{r_1} U_{2,x_0+\hat{1}r_1}^{r_2} U_{3,x_0+\hat{1}r_1+\hat{2}r_2}^{r_3} U_{4,x_0+\hat{1}r_1+\hat{2}r_2+\hat{3}r_3}^{r_4}, \quad (\text{A3})$$

<sup>30</sup> RG blocking of *free* one-component staggered fermions was discussed in Ref. [52].

where  $\hat{\mu}$  is the fine-lattice unit vector in the  $\mu$  direction. With the notation of Eq. (A2) we similarly have  $\bar{\psi}_{\alpha i}^{(0)}(\tilde{x}^{(0)}) = \bar{\chi} Q_{\alpha i}^{(0)\dagger}(\tilde{x}^{(0)})$ , where hermitian conjugation applies to the color matrices. Parallel transporting the fine-lattice variables entails well-defined transformation properties for the taste-basis variables under fine-lattice gauge transformations. Thus, gauge invariance is maintained by the blocking transformation.

The covariant blocking kernel (A2) illustrates, however, an inherent problem. Any concrete choice of the gauge-covariant blocking kernel will transform non-trivially under hypercubic rotations. In Eq. (A2) this is seen both in the special role of the hypercube's site with relative coordinates  $r_\mu = 0$ , because these relative coordinates transform non-trivially under hypercubic rotations; and, for a similar reason, in the specific ordering of traversing the axes in Eq. (A3).

The solution adopted in this paper is to perform the transition from the original one-component formalism to a taste representation as a gaussian RG-like transformation, in which no thinning out of the fermionic degrees of freedom (but only of the gauge field) occurs. For the taste-basis Dirac operator resulting from this transformation, see Ref. [15]. This comprises the special  $k = 0$  blocking step introduced in Sec. II A. Within the gaussian blocking transformation,  $\psi(\tilde{x})$  is loosely equal to  $Q^{(0)}(\tilde{x}^{(0)}; \mathcal{U}) \chi$ , and  $\bar{\psi}(\tilde{x})$  is loosely equal to  $\bar{\chi} Q^{(0)\dagger}(\tilde{x}^{(0)}; \mathcal{U})$ . For a precise statement, see App. B.

Of course, replacing the unitary change of variables (A2) by a gaussian transformation does not by itself solve the difficulty with hypercubic rotations. But we may now overcome it by creating a coherent superposition over a family of different blocking transformations. This is explained in App. E (see also App. D).

## 2. Fermion blocking kernels for $1 \leq k \leq n$

For completeness, let me specify the fermion blocking kernels of the subsequent,  $1 \leq k \leq n$  blocking steps. In the free theory I take

$$Q^{(k)}(\tilde{x}^{(k)}) \psi^{(k-1)} = \frac{1}{16} \sum_{r_\mu=0,1} \psi^{(k-1)}(2\tilde{x}^{(k)} + r). \quad (\text{A4})$$

In analogy with Eq. (A2) we may define a covariant version,

$$Q^{(k)}(\tilde{x}^{(k)}; \mathcal{V}^{(k-1)}) \psi^{(k-1)} = \frac{z^{(k)}}{16} \sum_{r_\mu=0,1} \mathcal{W}\left(2\tilde{x}^{(k)}, 2\tilde{x}^{(k)} + r; \mathcal{V}^{(k-1)}\right) \psi^{(k-1)}(2\tilde{x}^{(k)} + r), \quad (\text{A5})$$

where the parallel transporters are defined analogously to Eq. (A3), but now in terms of the blocked gauge field of the  $(k-1)^{\text{th}}$  lattice. These definitions imply that the linear transformation  $Q^{(0)}$  is unitary, whereas for  $1 \leq k \leq n$ , the product  $Q^{(k)} Q^{(k)\dagger}$  is equal to  $(z^{(k)})^2/16$  times the identity matrix on the  $k^{\text{th}}$  lattice. The difficulty with hypercubic symmetry recurs at every blocking step, and again it is solved in a similar manner (see App. E).

The constants  $z^{(k)}$  are adjusted to impose a wave-function renormalization condition on the fermion fields at each blocking level. Usually, lattice renormalization produces factors of  $\log(a\mu)$  where  $\mu$  is the renormalization scale. But in an RG-blocking setup one has  $a \rightarrow a_{k-1}$ ,  $\mu \rightarrow 1/a_k$  at the  $k^{\text{th}}$  blocking step. Whence  $\log(a\mu) \rightarrow \log(2)$ , and we may expect  $z^{(k)} = 1 + c_k g_r^2(a_k)/(16\pi^2)$ , where  $c_k = O(1)$ . Of course, the product of all the  $z^{(k)}$ 's can

diverge (or vanish) in the limit  $n \rightarrow \infty$ , as dictated by the integrated anomalous dimension of the fermion field.

The expectation value of any product of local composite operators constructed from the coarse-lattice fields will always be finite. Therefore, composite operators do not necessarily require a separate renormalization. (Once again this can be explained by the fact that the ratio of the cutoff and renormalization scales is a finite, fixed number.) One might, however, opt to impose specific renormalization conditions for certain composite operators. A renormalization condition imposed on a composite operator at the coarse-lattice scale will in general entail some finite renormalization.

### 3. Positivity of $\det(D_n)$ and $\det(G_n^{-1})$

Here I prove that  $\det(D_n)$  and  $\det(G_n^{-1})$  are positive for  $m > 0$ . In more detail, I will prove that, like  $\det(D_{stag})$ , also  $\det(D_n)$  is real and strictly positive for  $m > 0$ . It follows from Eq. (2.12) that  $\det(G_n^{-1})$  is real positive (for the issue of zero eigenvalues of  $G_n^{-1}$ , see Sec. IV B).

I begin by noting [15] that  $[\gamma_5 \otimes \xi_5] D_n [\gamma_5 \otimes \xi_5] = D_n^\dagger$ .<sup>31</sup> It follows that  $[\gamma_5 \otimes \xi_5] D_n$  is hermitian, and  $\det(D_n)$  is real. Moreover, complex eigenvalues of  $D_n$  must occur in pairs with conjugate values. Therefore  $\det(D_n)$  will be (strictly) positive if all the real eigenvalues are (strictly) positive.<sup>32</sup>

By “undoing” the all the blocking steps one can express the blocked propagator as

$$D_n^{-1} = R_n + Q_n D_{stag}^{-1} Q_n^\dagger, \quad (\text{A6})$$

where  $Q_n = Q^{(n)} Q^{(n-1)} \dots Q^{(0)}$ , and  $R_n > 0$  is determined iteratively from  $R_0 = \alpha_0^{-1}$  and  $R_k = \alpha_k^{-1} + ((z^{(k)})^2/16) R_{k-1}$  (for the free theory, see Eq. (H2)). It is now straightforward to show that, if  $\Psi$  is an eigenstate of  $D_n$  with real eigenvalue  $\lambda$ , then

$$\frac{1}{\lambda} = R_n + \Psi^\dagger Q_n \frac{m}{-D_{msls}^2 + m^2} Q_n^\dagger \Psi. \quad (\text{A7})$$

I have used that  $D_{stag} = D_{msls} + m$ , where the massless staggered operator  $D_{msls}$  is anti-hermitian. For  $m > 0$ , it follows that  $R_n \leq \lambda^{-1} < \infty$ . Hence all the real eigenvalues of  $D_n$  are (finite and) strictly positive for  $m > 0$ .

## APPENDIX B: MORE DETAILS ON THE PULL-BACK MAPPING

Here I discuss in more detail the pull-back mapping introduced in Sec. II A. First, considering the original as well as all the blocked gauge fields as a fixed background, let us discuss the pull-back mapping  $\mathcal{T}_F^{(j,n)}$  of the fermions only. In analogy with Eq. (2.14) it is defined for  $-1 \leq j \leq n-1$  by

$$\mathcal{T}_F^{(j,n)} \mathcal{O}^{(n)} = \int \prod_{k=j+1}^n [\mathcal{D}\psi^{(k)} \mathcal{D}\bar{\psi}^{(k)}] \exp \left[ - \sum_{k=j+1}^n \mathcal{K}_F^{(k)} \right] \mathcal{O}^{(n)}. \quad (\text{B1})$$

<sup>31</sup> The transformation (A1) implies  $\xi_\mu = \gamma_\mu^T$  if the Dirac and the taste matrices both act from the left.

<sup>32</sup> If  $D_n$  has no real eigenvalues, the strict positivity of  $\det(D_n)$  follows trivially.

As an example, consider the action of  $\mathcal{T}_F^{(n-1,n)}$  on a fermion bilinear. Using Eq. (B1) one has

$$\mathcal{T}_F^{(n-1,n)} \left[ \psi^{(n)}(\tilde{x}^{(n)}) \bar{\psi}^{(n)}(\tilde{y}^{(n)}) \right] = \frac{\delta_{\tilde{x}^{(n)}, \tilde{y}^{(n)}}}{\alpha_n} + \left[ Q^{(n)}(\tilde{x}^{(n)}) \psi^{(n-1)} \right] \left[ \bar{\psi}^{(n-1)} Q^{(n)\dagger}(\tilde{y}^{(n)}) \right]. \quad (\text{B2})$$

The resemblance to Eq. (2.10b) is evident. Notice that for  $\tilde{x}^{(n)} \neq \tilde{y}^{(n)}$  there is no contact term. This generalizes to the product of any number of fermion and anti-fermion fields. Thus, the fermion pull-back mapping realizes the operator relation

$$\mathcal{T}_F^{(n-1,n)} \psi^{(n)}(\tilde{x}^{(n)}) \simeq Q^{(n)}(\tilde{x}^{(n)}) \psi^{(n-1)}, \quad (\text{B3a})$$

$$\mathcal{T}_F^{(n-1,n)} \bar{\psi}^{(n)}(\tilde{x}^{(n)}) \simeq \bar{\psi}^{(n-1)} Q^{(n)\dagger}(\tilde{x}^{(n)}), \quad (\text{B3b})$$

where the right-hand sides were defined in App. A, and where the  $\simeq$  sign means equality up to the contact terms that arise when a fermion and an anti-fermion reside on the same site of the coarse lattice. Observe that, for the fermion kernels of App. A, if no fermion and anti-fermion reside on the same site of the coarse lattice, then no contact terms will arise under the pull-back  $\mathcal{T}_F^{(j,n)}$  for any  $j$ .

Next let us consider the action of the pull-back mapping on the gauge fields as well. First, a few more details on the gauge-field blocking kernels are needed. The non-linear blocking kernel  $\mathcal{B}_k$  is constructed as a sum over the links of the  $k^{\text{th}}$  lattice,

$$\mathcal{B}_k \left( \mathcal{V}^{(k)}, \mathcal{V}^{(k-1)} \right) = \sum_{\mu, \tilde{x}^{(k)}} \mathcal{F}_k \left[ V_{\mu, \tilde{x}^{(k)}}^{(k)}, W_{\mu, \tilde{x}^{(k)}}^{(k)} \left( \mathcal{V}^{(k-1)} \right) \right]. \quad (\text{B4})$$

A simple choice, consistent with the gauge-transformation properties of the fermion kernels (A5), is

$$\mathcal{F}_k(V, W) = -\beta_k \text{tr} (V^\dagger W), \quad (\text{B5a})$$

$$W_{\mu, \tilde{x}^{(k)}}^{(k)} = V_{\mu, 2\tilde{x}^{(k)}}^{(k-1)} V_{\mu, 2\tilde{x}^{(k)} + \hat{\mu}}^{(k-1)}, \quad (\text{B5b})$$

where  $\beta_k > 0$  is a new blocking parameter. Since  $W \in SU(3)$ , one can use the invariance of the Haar measure to show that  $\mathcal{N}_k(\mathcal{V}^{(k-1)})$  in Eq. (2.4b) reduces to a numerical constant. Many other choices of  $\mathcal{B}_k$  are possible, see e.g. Ref. [30].

Considering the defining equation (2.2a), the gauge-field blocking kernel  $\mathcal{B}_k$  may be viewed as a generalized action. This generalized action couples each  $k^{\text{th}}$ -lattice link  $V_{\mu, \tilde{x}^{(k)}}^{(k)}$  to the gauge field on the  $(k-1)^{\text{th}}$  lattice, but it does not couple directly any two  $k^{\text{th}}$ -lattice links. As an example, let  $F(g)$  denote some function of  $g \in SU(3)$ . Considering an operator of the form  $F(V_{\mu, \tilde{x}^{(n)}}^{(n)}) \mathcal{O}^{(n)}$ , where  $\mathcal{O}^{(n)}$  does not depend on  $V_{\mu, \tilde{x}^{(n)}}^{(n)}$ , it follows that

$$\mathcal{T}^{(n-1,n)} \left[ F \left( V_{\mu, \tilde{x}^{(n)}}^{(n)} \right) \mathcal{O}^{(n)} \right] = \left[ \mathcal{T}^{(n-1,n)} F \left( V_{\mu, \tilde{x}^{(n)}}^{(n)} \right) \right] \left[ \mathcal{T}^{(n-1,n)} \mathcal{O}^{(n)} \right]. \quad (\text{B6})$$

Returning to the general case, while the explicit expression gets more complicated with every pull-back step, the pull-back mapping is ultra-local because the blocking kernels are. If  $\mathcal{O}^{(n)}$  has a compact support, then the support of  $\mathcal{T}^{(j,n)} \mathcal{O}^{(n)}$  will only slightly increase for all  $j < n$ .



## APPENDIX C: ENSEMBLES OF BLOCKED CONFIGURATIONS

Having introduced all the blocking kernels (see App. A and App. B), let us discuss the generation of ensembles of blocked gauge fields. This is necessary, for example, for the computation of the coarse-lattice observables numerically. The issue is how to generate blocked gauge-field configurations from pre-existing fine-lattice configurations. Using Eq. (2.12) in Eq. (4.2) for  $t = 1$ , we get

$$Z_n(1, n_r, a_c) = \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] \exp\left(-S_g - \sum_{k=0}^n \mathcal{K}_B^{(k)}\right) \det^{n_r}(D_{stag}) \quad (\text{C1a})$$

$$= \int \mathcal{D}\mathcal{U} \exp(-S_g) \det^{n_r}(D_{stag}) \equiv Z(n_r). \quad (\text{C1b})$$

Equation (C1b) reminds us that, in the original staggered theory, the Boltzmann weight of the fine-lattice gauge field  $\mathcal{U}$  has the form  $\exp(-S_g) \det^{n_r}(D_{stag})$  with  $n_r = \frac{1}{4}, \frac{1}{2}$ , or 1. Equation (C1a) provides a Boltzmann weight for all the blocked gauge fields as well. In view of Eq. (C1b), the process begins with an ensemble of fine-lattice configurations generated in the usual way. Given a “mother” configuration  $\mathcal{U}_i$  in this ensemble, one can generate a “daughter” configuration of the once-blocked gauge field  $\mathcal{V}_i^{(0)}$  by a new Monte-Carlo process, by taking  $\exp[-\mathcal{K}_B^{(0)}(\mathcal{V}_i^{(0)}, \mathcal{U}_i)]$  as a Boltzmann weight while holding the fine-lattice gauge field  $\mathcal{U}_i$  fixed. As it should, the probability to obtain the pair  $\{\mathcal{U}_i, \mathcal{V}_i^{(0)}\}$  is given by the product of the original (normalized) Boltzmann weight  $Z^{-1}(n_r) \exp(-S_g(\mathcal{U}_i)) \det^{n_r}(D_{stag}(\mathcal{U}_i))$  and the new Boltzmann weight  $\exp[-\mathcal{K}_B^{(0)}(\mathcal{V}_i^{(0)}, \mathcal{U}_i)]$ . The process may be repeated on further blocking steps  $1 \leq k \leq n$ , each time generating a  $k^{\text{th}}$ -lattice daughter configuration from the existing  $(k-1)^{\text{th}}$ -lattice daughter configuration  $\mathcal{V}_i^{(k-1)}$  using the Boltzmann weight  $\exp[-\mathcal{K}_B^{(k)}(\mathcal{V}_i^{(k)}, \mathcal{V}_i^{(k-1)})]$ .

Once an ensemble of fine-lattice configurations and of daughter configurations for all  $0 \leq k \leq n$  has been generated, it can be used to calculate any observable. The blocked-lattice fermion propagator  $D_n^{-1}$  can be calculated by repeatedly applying Eq. (2.10) until an expression involving the fine-lattice propagator  $D_{stag}^{-1}$  is obtained. The blocking kernel  $Q^{(k)}$  is an explicit functional of  $\mathcal{V}^{(k-1)}$  only, therefore it should be evaluated using a  $(k-1)^{\text{th}}$ -lattice daughter configuration.<sup>33</sup>

## APPENDIX D: LATTICE SYMMETRIES UNDER THE BLOCKING TRANSFORMATION

As explained in Sec. II C, thanks to the pull-back mapping each coarse-lattice observable is at the same time a fine-lattice observable; as such, it is constrained by all the staggered-fermion symmetries of the original theory. In a more technical sense, a given fine-lattice

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<sup>33</sup> In the limit  $\beta_k \rightarrow \infty$  the blocking Boltzmann weight  $\exp[-\mathcal{K}_B^{(k)}(\mathcal{V}^{(k)}, \mathcal{V}^{(k-1)})]$  collapses to a  $\delta$ -function, and the daughter configurations reduce to well-defined functionals of the original fine-lattice gauge field. In the case of Eq. (B5), for example, one finds that that  $V_{\mu, \tilde{x}^{(0)}}^{(0)}$  is equal to  $U_{\mu, 2\tilde{x}^{(0)}} U_{\mu, 2\tilde{x}^{(0)} + \hat{\mu}}$ , and  $V_{\mu, \tilde{x}^{(k)}}^{(k)}$  is equal to  $V_{\mu, 2\tilde{x}^{(k)}}^{(k-1)} V_{\mu, 2\tilde{x}^{(k)} + \hat{\mu}}^{(k-1)}$ .

symmetry may or may not survive as a manifest symmetry on the coarse lattice. I will now discuss the fine-lattice symmetries one by one.

Translation and gauge invariance have a prominent role, and they are secured by construction. With the fermion and gauge-field blocking kernels introduced above, the blocked-lattice action  $S_n$  (cf. Eq. (2.2b)) retains these symmetries manifestly.<sup>34</sup>

The situation is more subtle with respect to hypercubic symmetry. As explained in App. A, the fermion blocking kernels transform non-trivially under  $90^\circ$  rotations. As a result, with the blocking transformations as introduced in Eq. (2.2), in fact  $S_n$  is not invariant under hypercubic rotations. On a closer look, the reason can also be understood as follows. Consider the pull-back  $\mathcal{T}^{(-1,n)}\mathcal{O}^{(n)}$  of some operator from the coarse to the fine lattice. Under a fine-lattice rotation,  $\mathcal{T}^{(-1,n)}\mathcal{O}^{(n)}$  transforms in the usual way. But, because of the non-trivial transformation properties of the blocking kernels, the rotated fine-lattice operator cannot be obtained as the pull-back of any coarse-lattice operator! In other words, the observables of the coarse-lattice theory do not constitute complete representations of the hypercubic group.

This difficulty can be solved by allowing the blocking kernels to depend on additional degrees of freedom, or *disorder fields*. Each blocking step in Eq. (2.2) is promoted to a coherent superposition of block transformations summed over all values of the disorder fields. The details are given in App. E. Briefly, a disorder field allows for parallel transporting of the fermion variables of a given  $2^4$  hypercube to any of its sixteen sites in turn. Another disorder field allows for all possible orderings for traversing the axes. The gauge-field blocking kernels are similarly adapted. With the disorder fields in place, the blocked action becomes manifestly invariant under hypercubic rotations.

Next, the  $U(1)_\epsilon$  symmetry of the (massless) one-component formalism turns into a Ginsparg-Wilson-Lüscher (GWL) chiral symmetry [26, 43] in the blocked-lattice theories. See Refs. [15, 20] for a detailed discussion of both the massless and the massive cases. For some further observations, see App. E.

The last symmetry of the (one-flavor) staggered theory is shift symmetry. It is generated by four anti-commuting elements  $L_\mu$ . The action of  $L_\mu$  involves a one-unit translation in the  $\mu$  direction, as well as the multiplication of  $\chi(x)$  and  $\bar{\chi}(x)$  by sign factors. In the low energy limit, shift symmetry reduces to a discrete subgroup of taste- $U(4)$ . The importance of shift symmetry is that, without it, a cutoff-scale mass term that breaks the  $U(4)$  taste symmetry may be induced. This unacceptable mass term is indeed generated if one couples the free taste-basis Dirac operator directly to a gauge field [54], as was found by an explicit one-loop calculation in Ref. [53].

In contrast, the taste-basis representation constructed in this paper avoids this problem. To see this, consider the expectation value of the pulled-back fermion propagator (compare Eq. (A6))<sup>35</sup>

$$G(x^{(n)}, y^{(n)}) = \left\langle \mathcal{T}^{(-1,n)} \left[ \psi^{(n)}(x^{(n)}) \bar{\psi}^{(n)}(y^{(n)}) \right] \right\rangle_{-1}. \quad (\text{D1})$$

Here, cf. Eq. (2.15), the subscript “ $-1$ ” refers to the expectation value of the pulled-back operator in the original staggered theory. It is straightforward to verify that the presence of

<sup>34</sup> Obviously, the size of the translation group gets smaller with each blocking step.

<sup>35</sup> We may either consider the expectation value in Eq. (D1) in a fixed gauge, or replace it by a gauge invariant one, obtained e.g. by connecting the coarse-lattice fermion and anti-fermion by a coarse-lattice Wilson line.

the offensive mass term in the coarse-lattice propagator would imply that  $G(x^{(n)}, y^{(n)})$  is not invariant under shift symmetry [8, 32]. This is impossible, however, because  $G(x^{(n)}, y^{(n)})$  is a correlation function of the original staggered theory.

As a last exercise that nicely exhibits how the coarse-lattice observables are constrained by the fine-lattice symmetries, let us examine the two-point function of a coarse-lattice operator with the quantum numbers of an exactly massless (taste non-singlet) Goldstone pion [20], in a *two*-flavor theory. As an interpolating field we may take

$$\pi_{ab}^{(n)}(\tilde{x}^{(n)}) = \overline{\psi}_a^{(n)}(\tilde{x}^{(n)}) [\gamma_5 \otimes \xi_5] \psi_b^{(n)}(\tilde{x}^{(n)}), \quad (\text{D2a})$$

where again  $\gamma_5$  and  $\xi_5$  act on the Dirac and the taste indices respectively, and  $a, b = 1, 2$ , label the staggered flavor. Then

$$\left\langle \pi_{12}^{(n)}(\tilde{x}^{(n)}) \pi_{21}^{(n)}(\tilde{y}^{(n)}) \right\rangle_n = \langle \mathcal{G}(\tilde{x}^{(n)}, \tilde{y}^{(n)}) \rangle, \quad (\text{D2b})$$

where

$$\mathcal{G}(\tilde{x}^{(n)}, \tilde{y}^{(n)}) = \text{tr} \left( [\gamma_5 \otimes \xi_5] D_n^{-1}(\tilde{x}^{(n)}, \tilde{y}^{(n)}) [\gamma_5 \otimes \xi_5] D_n^{-1}(\tilde{y}^{(n)}, \tilde{x}^{(n)}) \right). \quad (\text{D2c})$$

The expectation value on the left-hand side of Eq. (D2b) is with respect to the partition function in Eq. (2.2), while on the right-hand side it is with respect to Eq. (C1a).<sup>36</sup> Via the pull-back mapping, the interpolating coarse-lattice fields we use represent specific smeared sources on the fine lattice. Now, one can show that  $\mathcal{G}(\tilde{x}^{(n)}, \tilde{y}^{(n)})$  is strictly positive [15, 20]. This rules out the possibility of destructive interference caused by these smeared sources; the asymptotic decay rate of the correlator must be dictated by the lightest excitation of the original staggered theory in that channel, the Goldstone pion. Once again, this shows that no fermion mass terms that contradict any of the symmetries of the original staggered theory could be generated, because such a mass term would completely change the long-distance behavior of this correlator.

A feature that may be confusing on first acquaintance is that the limiting  $n \rightarrow \infty$  coarse-lattice theory is invariant only under  $90^\circ$  rotations, and not under continuous rotations. This can be understood as follows. Via the pull-back mapping, even a nominally scalar (or pseudoscalar) operator on the coarse lattice has in effect some internal structure for its support, that “remembers” the orientations of the axes of the lattice. The lack of manifest invariance under continuous rotations in the coarse-lattice theory is, once again, because its observables do not constitute complete representations of this symmetry group.<sup>37</sup> In a formal sense, the continuum-limit observables accessible by the coarse-lattice theory form a discrete subset of the set of “all” continuum-limit observables. If we would keep decreasing the coarse-lattice spacing, we expect that the breaking of continuous rotational invariance should go to zero like some positive power of  $a_c \Lambda$ .

In more detail, consider as an example the  $n \rightarrow \infty$  limit of the pion two-point function in Eq. (D2b),

$$\bar{\mathcal{G}}(x, y; a_c) = \lim_{n \rightarrow \infty} \left\langle \pi_{12}^{(n)}(\tilde{x}) \pi_{21}^{(n)}(\tilde{y}) \right\rangle_n. \quad (\text{D3})$$

<sup>36</sup> In other words, the right-hand side is to be evaluated on an ensemble of blocked configurations, cf. App. C.

<sup>37</sup> In the free theory, one can check that the operator  $D_{rg}$  obtained in the  $n \rightarrow \infty$  limit (Eq. (2.18)) indeed has only hypercubic rotation invariance [20].

At large (euclidean) distances, where the correlator is dominated by the Goldstone-pion intermediate state, one expects the factorization

$$\bar{\mathcal{G}}(x, y; a_c) \approx e^{-m_\pi |x-y|} \mathcal{F}^2(n_\mu, a_c m_\pi). \quad (\text{D4})$$

Here  $|x - y|$  is the usual euclidean distance, and  $n_\mu$  is the unit vector pointing in the direction of  $x - y$ . Power corrections that depend on  $|x - y|$  have been suppressed. The (direction dependent!) form factor  $\mathcal{F}(n_\mu, a_c m_\pi)$  accounts for the coupling of our coarse-lattice interpolating field to the pion intermediate state it creates. The smeared fine-lattice operator that corresponds (via the pull-back mapping) to the coarse-lattice interpolating field is manifestly invariant under  $90^\circ$  rotations only. As noted above, in the limit  $a_c m_\pi \rightarrow 0$ , the form factor  $\mathcal{F}(n_\mu, a_c m_\pi)$  should approach an ( $n_\mu$ -independent) constant.

## APPENDIX E: HYPERCUBIC SYMMETRY AND DISORDER FIELDS

Here I discuss how manifest hypercubic invariance is recovered by summing over disorder fields at each blocking step. I will discuss mainly the  $k = 0$  step, which produces the transition from the one-component basis to the taste basis. Subsequent blocking steps work essentially in the same way, except that they are somewhat simpler because the subsequent blocking kernels act trivially on the Dirac and taste indices. Within this Appendix, I will usually use the term “coarse lattice” for the taste-basis lattice obtained via the  $k = 0$  blocking step, in which case I will drop the correspond superscript-label of the fields and the coordinates.

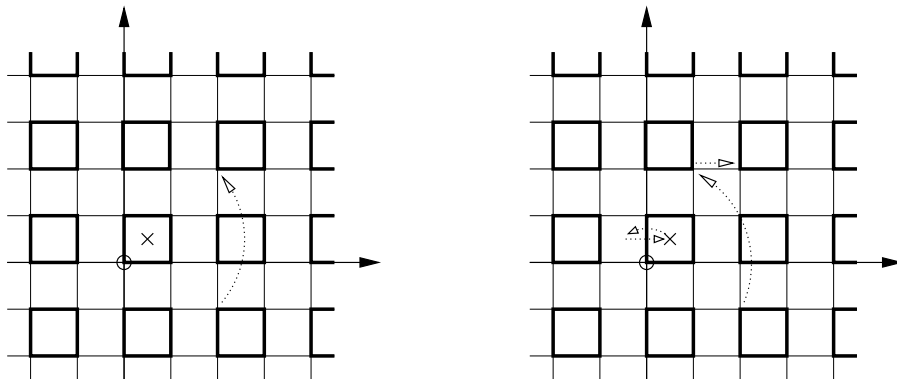


FIG. 1: *Rotations.* The two-dimensional example shows how fine- and coarse-lattice rotations are related. The small circle marks the origin. The point  $X = (\frac{1}{2}, \frac{1}{2})$  is marked by a cross. Thick squares show the blocking pattern. *Left panel:* Counter-clockwise  $90^\circ$  rotation about the point  $X$ . The blocked squares are mapped onto themselves: centers are mapped to centers; corners undergo a rotation with respect to the square’s center. *Right panel:* The same effect is achieved by a rotation about the origin, followed by a translation that brings the point  $X$  back into its original position.

The guiding principle is that we want to preserve the embedding of the coarse (taste-basis) lattice into the original fine lattice, *i.e.* we want to maintain the same breakup of the fine

lattice into  $2^4$  hypercubes. On the coarse lattice, the  $90^\circ$  rotation will be around the origin. As can be seen from Fig. 1, this corresponds to a fine-lattice rotation around the origin which is either followed by, or preceded with, a one-unit translation. Recall that, for staggered fermions, the fine-lattice translation group is generated by the four anti-commuting shifts  $L_\mu$  that involve a one-unit translation and the multiplication of the staggered fields by sign factors. Let us denote by  $\mathcal{R}$  the matrix that produces a  $90^\circ$  rotation around the origin. This defines a linear, homogeneous mapping of the coordinates, as well as of four-vectors. The coarse- and fine-lattice coordinates rotations are given by

$$\tilde{x} \rightarrow \tilde{x}' = \mathcal{R}\tilde{x}, \quad (\text{E1a})$$

$$x \rightarrow x' = \hat{\mathcal{R}}(x) \equiv \mathcal{R}x + \Delta, \quad (\text{E1b})$$

where  $\Delta$  is the one-unit translation that follows the rotation (Fig. 1). The inverse of Eq. (E1b) is

$$\hat{\mathcal{R}}^{-1}(x') = \mathcal{R}^{-1}(x' - \Delta) = \mathcal{R}^{-1}x' - \mathcal{R}^{-1}\Delta. \quad (\text{E2})$$

Making the vector index explicit, the fine-lattice rotation is given by  $x'_\mu = \mathcal{R}_{(\sigma\tau)\mu\nu} x_\nu + \delta_{\sigma\mu}$ . The rotation matrix  $\mathcal{R}_{(\sigma\tau)\mu\nu} = \delta_{\sigma\nu}\delta_{\tau\mu} - \delta_{\sigma\mu}\delta_{\tau\nu} + P_{(\sigma\tau)\mu\nu}$  produces the “counter-clockwise” rotation in the  $(\sigma, \tau)$  plane, whereby  $x'_\tau = x_\sigma$  and  $x'_\sigma = -x_\tau$ . Here  $P_{(\sigma\tau)\mu\nu} = \delta_{\mu\nu} - \delta_{\sigma\mu}\delta_{\sigma\nu} - \delta_{\tau\mu}\delta_{\tau\nu}$  is the projector on the  $d - 2$  invariant coordinates. For the same rotation one has  $\Delta = \hat{\sigma}$  in Eq. (E1b), namely, the follow-up translation is in the positive  $\hat{\sigma}$  direction.

Space-time transformations act on fields by prescribing their value at a point in terms of their value at the source of that point (under the “active” coordinates transformation). For a coarse-lattice rotation around the origin, the value of the transformed staggered field at a fine-lattice point  $x'$  will be determined in terms of its value at  $\hat{\mathcal{R}}^{-1}(x')$ . Therefore, the transformation applied to the staggered field is first a shift from  $x'$  back to  $x' - \Delta$ , and then a fine-lattice rotation (around the origin) back to the original orientation, cf. Eq. (E2). Performing this combined transformation using the rules given in Ref. [8] and plugging the result into the right-hand side of Eq. (A1) yields the taste-basis transformation rule for hypercubic rotations [53]

$$\psi(\tilde{x}) \rightarrow \psi'(\tilde{x}) = [R \otimes T^T] \psi(\mathcal{R}^{-1}\tilde{x}). \quad (\text{E3})$$

Here  $R_{(\sigma\tau)} = 2^{-1/2}(1 - \gamma_\sigma\gamma_\tau)$  is the usual Dirac rotation, while  $T_{(\sigma\tau)} = 2^{-1/2}(\gamma_\sigma - \gamma_\tau) = T_{(\sigma\tau)}^\dagger$  produces the rotation on the taste index.<sup>38</sup>

The fermion blocking kernels (A2) and (A5) are already gauge covariant. I now discuss how to “covariantize” their transformation properties under hypercubic rotations. The idea is to simply let any element of the fine lattice, be it a site or a link, transform as it should under the above fine-lattice rotation. For the fermion blocking kernels we need to make two choices. What choice is being made will be prescribed by a set of discrete-valued “disorder fields,” that reside on suitable elements of the coarse lattice. In detail, we have to decide to which one of the hypercube’s sixteen sites will all the fermion variables be parallel transported. The chosen site will be determined by a vector field  $\rho_\mu$ . The possible values of  $\rho_\mu(\tilde{x})$  are zero or one, and the fine-lattice coordinates of the chosen site will be  $2\tilde{x} + \rho(\tilde{x})$ . We must also decide in which order to traverse the axes. The ordering will be determined

<sup>38</sup> The transformation rule in Ref. [53] looks slightly different due to a further change of basis. See also footnote 31.

by another coarse-lattice field  $\hat{\omega} = \hat{\omega}(\tilde{x})$  that takes values in  $S_4$ , the permutation group of four elements.

Let us next establish the transformation rules of these fields. For the chosen-site field we demand that, if  $x = 2\tilde{x} + \rho(\tilde{x})$ , then this relation will be respected by the rotation. With  $x'$  and  $\tilde{x}'$  given by (E1), we must require  $x' = 2\tilde{x}' + \rho'(\tilde{x}')$ . It is straightforward to show that the required transformation rule is

$$\rho'(\tilde{x}') = \mathcal{R} \rho(\mathcal{R}^{-1}\tilde{x}') + \Delta. \quad (\text{E4})$$

Let us verify that Eq. (E4) is a consistent transformation on this field. We must verify that for  $\rho_\mu = 0, 1$ , also  $\rho'_\mu$  takes only these two values. Let us again consider the counter-clockwise rotation in the  $(\sigma, \tau)$  plane. Only the  $\sigma$  and  $\tau$  components undergo a non-trivial transformation, which reads explicitly  $\rho'_\tau(\mathcal{R}_{(\sigma\tau)}\tilde{x}) = \rho_\sigma(\tilde{x})$  and  $\rho'_\sigma(\mathcal{R}_{(\sigma\tau)}\tilde{x}) = -\rho_\tau(\tilde{x}) + 1$ . We see that the translation by the unit vector  $+\hat{\sigma}$  acts precisely to bring  $\rho'_\sigma(\mathcal{R}_{(\sigma\tau)}\tilde{x})$  back into the allowed range.

In order to write down the transformation rule for the axes-ordering field, let us use the defining representation of the permutation group  $S_4$  in terms of four-by-four orthogonal matrices, each of which has one entry equal to one and the rest equal to zero on every row or column. The axes ordering is then given by letting this matrix act on the constant four-vector  $v = (1234)$ , that is, act on the four-vector whose entries are given by  $v_\mu = \mu$ . With this, the transformation rule is

$$\hat{\omega}'(\tilde{x}') = \hat{\omega}(\mathcal{R}^{-1}\tilde{x}') \hat{\pi}(\mathcal{R}), \quad (\text{E5})$$

where the permutations  $\hat{\pi}(\mathcal{R}_{(\sigma\tau)}) \in S_4$  is represented by the four-by-four matrix  $\hat{\pi}(\mathcal{R}_{(\sigma\tau)})_{\mu\nu} = \delta_{\sigma\nu}\delta_{\tau\mu} + \delta_{\sigma\mu}\delta_{\tau\nu} + P_{(\sigma\tau)\mu\nu}$ .

For completeness, recall the transformation rule of the fine-lattice gauge field, which is conveniently expressed as [8]

$$U(x, y) \rightarrow U'(x, y) = U(\mathcal{R}^{-1}x, \mathcal{R}^{-1}y), \quad (\text{E6})$$

where

$$U(x, y) = \begin{cases} U_{\mu,x}, & y = x + \hat{\mu}, \\ U_{\mu,y}^\dagger, & y = x - \hat{\mu}, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{E7})$$

We are now ready to introduce new parallel transporters that transform covariantly under rotations. Let

$$\mathcal{W}(x, y, \hat{\pi}; \mathcal{U}), \quad (\text{E8})$$

be the parallel transporter from  $y$  back to  $x$ , which traverses the axes in the order determined by  $\hat{\pi} \in S_4$ , as follows. With the constant four vector  $v$  introduced above, we let  $v_{\hat{\pi}} = \hat{\pi}v$ . Starting at  $y$  and letting  $\nu = v_{\hat{\pi}}(4)$ , we first go along the  $\nu^{\text{th}}$  axis until the  $\nu^{\text{th}}$  coordinate is equal to  $x_\nu$ . The direction is determined by the sign of  $y_\nu - x_\nu$ . Then we go along the axis specified by  $v_{\hat{\pi}}(3)$  and so on. In four steps, each involving a straight line, we go from  $y$  back to  $x$ . Note that the parallel transporter in Eq. (A3) corresponds to the special case of choosing  $\hat{\pi}$  as the identity element.

Armed with the more general parallel transporter (E8), we modify the fermion kernel of the  $k = 0$  step by replacing  $\mathcal{W}$  of Eq. (A3) with  $\mathcal{W}(2\tilde{x} + \rho(\tilde{x}), 2\tilde{x} + r(\tilde{x}), \hat{\omega}(\tilde{x}); \mathcal{U})$ . For convenience, the dummy summation variable of Eq. (A2) has been promoted to a field; its

transformation properties are, obviously, the same as those of  $\rho(\tilde{x})$ . The so-constructed parallel transporter transforms as

$$\begin{aligned} \mathcal{W}(2\tilde{x} + \rho(\tilde{x}), 2\tilde{x} + r(\tilde{x}), \hat{\omega}(\tilde{x}); \mathcal{U}) &\rightarrow \\ &\rightarrow \mathcal{W}(2\tilde{x} + \rho'(\tilde{x}), 2\tilde{x} + r'(\tilde{x}), \hat{\omega}'(\tilde{x}); \mathcal{U}') \\ &= \mathcal{W}(2\mathcal{R}^{-1}\tilde{x} + \rho(\mathcal{R}^{-1}\tilde{x}), 2\mathcal{R}^{-1}\tilde{x} + r(\mathcal{R}^{-1}\tilde{x}), \hat{\omega}(\mathcal{R}^{-1}\tilde{x}); \mathcal{U}). \end{aligned} \quad (\text{E9})$$

With this, the right-hand side of Eq. (A2) attains the same hypercubic transformation properties as the taste-basis field, cf. Eq. (E3).

Now that the fermions are each time parallel transported to a different hypercube's site, we must also modify the gauge-field blocking kernels, so as to maintain gauge invariance of the coarse-lattice theory. For the  $k = 0$  step, the new gauge-field blocking kernel is obtained by replacing Eq. (B5b) with

$$W_{\mu, \tilde{x}} = \mathcal{W}(2\tilde{x} + \rho(\tilde{x}), 2(\tilde{x} + \hat{\mu}) + \rho(\tilde{x} + \hat{\mu}), \hat{\Omega}_\mu(\tilde{x}); \mathcal{U}), \quad (\text{E10})$$

where again  $\mathcal{W}$  is defined by Eq. (E8). In Eq. (E10), the axes ordering is chosen independently for each coarse-lattice link, according to a new disorder field  $\hat{\Omega}_\mu(\tilde{x})$  taking values in the permutation group  $S_4$ . Introducing notation analogous to Eq. (E7),

$$\hat{\Omega}(\tilde{x}, \tilde{y}) = \begin{cases} \hat{\Omega}_\mu(\tilde{x}), & \tilde{y} = \tilde{x} + \hat{\mu}, \\ \hat{\Omega}_\mu(\tilde{y}), & \tilde{y} = \tilde{x} - \hat{\mu}, \\ 0, & \text{otherwise,} \end{cases} \quad (\text{E11})$$

(note that  $\hat{\Omega}_\mu(\tilde{x}) = \hat{\Omega}_\mu^\dagger(\tilde{x})$  in this case), its transformation rule is (compare Eq. (E5))

$$\hat{\Omega}(x, y) \rightarrow \hat{\Omega}'(x, y) = \hat{\Omega}(\mathcal{R}^{-1}\tilde{x}, \mathcal{R}^{-1}\tilde{y}) \hat{\pi}(\mathcal{R}). \quad (\text{E12})$$

We are now ready for the implementation. A complete set of disorder fields is introduced at each blocking step. Re-instating the blocking-step label, the “measure” for the disorder fields is

$$\sum_{(k)} \equiv \prod_{\tilde{x}^{(k)}} \left[ \frac{1}{24} \sum_{\hat{\omega}^{(k)}(\tilde{x}^{(k)}) \in S_4} \right] \prod_{\mu, \tilde{x}^{(k)}} \left[ \frac{1}{16 \cdot 24} \sum_{\rho_\mu^{(k)}(\tilde{x}^{(k)})=0,1} \sum_{\hat{\Omega}_\mu^{(k)}(\tilde{x}^{(k)}) \in S_4} \right]. \quad (\text{E13})$$

The  $k = 0$  blocking step takes the form

$$Z = \int \mathcal{D}\mathcal{U} \mathcal{D}\chi \mathcal{D}\bar{\chi} \exp[-S_g(\mathcal{U}) - \bar{\chi} D_{stag}(\mathcal{U}) \chi] \quad (\text{E14a})$$

$$\begin{aligned} &\times \sum_{(0)} \int \mathcal{D}\mathcal{V}^{(0)} \mathcal{D}\psi^{(0)} \mathcal{D}\bar{\psi}^{(0)} \exp \left[ -\mathcal{B}_0(\mathcal{V}^{(0)}, \mathcal{U}) - \mathcal{N}_0(\mathcal{U}) \right] \\ &\times \exp \left[ \alpha_0 \left( \bar{\psi}^{(0)} - \bar{\chi} Q^{(0)\dagger}(\mathcal{U}) \right) \left( \psi^{(0)} - Q^{(0)}(\mathcal{U}) \chi \right) \right] \end{aligned} \quad (\text{E14b})$$

$$= \int \mathcal{D}\mathcal{V}^{(0)} \mathcal{D}\psi^{(0)} \mathcal{D}\bar{\psi}^{(0)} \exp \left[ -S_0(\mathcal{V}^{(0)}, \psi^{(0)}, \bar{\psi}^{(0)}) \right]. \quad (\text{E14c})$$

The blocking transformation is introduced on line (E14b). As promised, it consists of a coherent superposition of “elementary” blocking transformations, each corresponding to a

particular set of values of all the disorder fields. In going from Eq. (E14b) to Eq. (E14c) we both integrate out the original fields, and sum over all values of the disorder fields. Similar coherent superpositions of blocking transformations are introduced in all subsequent steps.

With all the disorder fields in place, the coarse-lattice theory obtained at the  $n^{\text{th}}$  step is manifestly invariant under hypercubic rotations. To see this, observe that all the blocking kernels in Eq. (2.3) become hypercubic-rotation invariant thanks to the transformation properties endowed to the disorder fields. The original action is invariant too, and the sum of the original action plus the blocking kernels may be regarded as a generalized action, which is hypercubic-rotation invariant as well. The effective action  $S_n$  obtained after integrating out any number of fields retains the same invariance.

Because the values of the disorder *fields* can vary locally, the resulting action  $S_n$  can be written as a sum over coarse-lattice sites of a local hypercubic scalar. The same would not be true had we restricted the disorder fields to take globally constant values only. Such a global sum would amount to averaging correlation functions of different (blocked) theories, and the result would in general violate clustering. If (manifest) hypercube symmetry was enforced by global averaging, violation of clustering would occur at every blocking level  $k$ , where it is expected to scale like a power of  $a_k$  (and, ultimately, like a power of  $a_c$ ), which is unacceptable. This unpleasant situation is avoided, however, because the disorder fields are local fields.

Turning to the representation (2.8), in order to maintain the gaussian nature of the remaining fermion integral one must refrain from integrating out any fields other than fermions. This means that summations over the disorder fields must not be carried out explicitly as well. The representation then takes the form

$$Z = \sum_{(0)} \sum_{(1)} \cdots \sum_{(n)} \int \mathcal{D}\mathcal{U} \prod_{k=0}^n [\mathcal{D}\mathcal{V}^{(k)}] \mathbf{B}_n(1; \mathcal{U}, \{\mathcal{V}^{(k)}\}) \times \int \mathcal{D}\psi^{(n)} \mathcal{D}\bar{\psi}^{(n)} \exp\left(-\bar{\psi}^{(n)} D_n \psi^{(n)}\right), \quad (\text{E15})$$

Related subsequent equations (e.g. in Sec. II E) are modified accordingly. We should also modify the process of generating ensembles of blocked-lattice configurations. Integrating out the remaining fermion fields as well as all the blocked gauge fields in Eq. (E15) we obtain

$$Z = \sum_{(0)} \sum_{(1)} \cdots \sum_{(n)} \int \mathcal{D}\mathcal{U} \exp(-S_g) \det(D_{stag}), \quad (\text{E16})$$

which is to be compared with Eq. (C1b).<sup>39</sup> This equation states the (obvious) result that, with no more blocked gauge fields around, the disorder fields decouple from the original theory. Therefore, the original fine-lattice gauge field is to be generated as always with its usual Boltzmann weight  $Z^{-1} \exp(-S_g) \det(D_{stag})$ , while all the disorder fields are to be generated with a flat measure. Any “tensor-product” configuration made of a fine-lattice gauge-field configuration and a configuration of all the disorder fields then serves as a mother configuration for the production of the chain of daughter configurations of the blocked gauge fields. In practice, this entails the simple instruction that a new set of values of the disorder fields is to be picked up at random for any new evaluation of a blocking kernel.

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<sup>39</sup> The generalization to  $n_r \neq 1$ , cf. Eq. (C1), is straightforward.



Last let me address the following question. The fermion blocking kernels are not invariant under ordinary chiral transformations, and this leads directly to the GW relation, and to the replacement of any ordinary chiral symmetry by its GWL cousin, as discussed for the case at hand in Refs. [15, 20]. In comparison to hypercubic rotations, the invariance (of the massless limit) under ordinary chiral symmetries is in fact lost already in the free theory after one or more blocking steps. One may wonder whether a similar trick with some new disorder fields would help us retain the invariance under ordinary chiral transformations in the blocked theory. The answer is yes, but it carries with it very little gain as I will now explain.

In the free theory, the generator of (ordinary) chiral transformations in the taste basis is  $[\gamma_5 \otimes \xi_5]$ . Within the blocking process, we may enforce the invariance under the same (global) chiral symmetry by augmenting each of the fermion blocking kernels with a new disorder field  $M^{(k)} = M^{(k)}(\tilde{x}^{(k)})$  transforming like a mass spurion. Specifically,  $M^{(k)}(\tilde{x}^{(k)})$  is a sixteen by sixteen matrix, labeled by a double, Dirac and taste, index. It takes values in the  $U(1)$  group  $\exp(i\theta[\gamma_5 \otimes \xi_5])$ . Again taking the  $k = 0$  step as an example, the new blocking kernel would take the form

$$\mathcal{K}_F^{(0)} = \alpha_0 \left( \overline{\psi}^{(0)} - \overline{\chi} Q^{(0)\dagger}(\mathcal{U}) \right) M^{(0)} \left( \psi^{(0)} - Q^{(0)}(\mathcal{U})\chi \right), \quad (\text{E17})$$

where the dependence on all other disorder fields has been suppressed. Under a global  $U(1)$  chiral rotation, and assuming that the taste-basis field transforms as  $\psi \rightarrow \exp(i\theta[\gamma_5 \otimes \xi_5])\psi$ , the new disorder field transforms as

$$M^{(k)}(\tilde{x}^{(k)}) \rightarrow \exp(-2i\theta[\gamma_5 \otimes \xi_5]) M^{(k)}(\tilde{x}^{(k)}). \quad (\text{E18})$$

Taken alone, the new blocking kernel (E17) is in fact invariant not only under global but also under local  $U(1)$  chiral transformations. However, the original action is only invariant under the corresponding global transformations (the  $U(1)_\epsilon$  symmetry) in the massless limit. Hence, the blocked action obtained after integrating over the original staggered fields, as well as over all values of the new disorder field, will be invariant under global chiral rotations only, as it should. Of course, that blocked action is not bilinear in the fermion fields, nor can it be reasonably approximated by any bilinear fermion action even in the free case. This has to be so, or else the Nielsen-Ninomiya theorem [55] would be violated.

More relevant is the role of the chiral disorder fields within the representation (E15). Particularly illuminating is to consider their effect on the fermions pull-back mapping (B1). It is easily seen that the role of  $M^{(k)}(\tilde{x}^{(k)})$  is to multiply the contact term in Eq. (B2), obtained while undoing the  $k^{\text{th}}$  blocking step, by a chiral phase which depends on the Dirac and the taste indices. Under a “complete” fermion pull-back all the way to the staggered theory on the original fine lattice, nothing would depend on the chiral disorder fields, apart from the contact terms encountered along the way. The integration at each blocking step over all values of  $M^{(k)}(\tilde{x}^{(k)})$  for all  $\tilde{x}^{(k)}$  would thus wipe out all the contact terms.

For non-coinciding points  $\tilde{x}^{(n)} \neq \tilde{y}^{(n)}$  on the “last” coarse lattice, however, contact terms are absent anyway. The upshot is that, when evaluating blocked fermion propagators on blocked ensembles (cf. App. C), we have the following choice for coinciding coarse-lattice points. We may either evaluate the contact terms generated by the fermions pull-back mapping, assuming there were no chiral disorder fields; or else, we are free to drop them, assuming that these disorder fields were present. Whichever choice we make, it is of no consequence for any non-coinciding coarse-lattice points. The coarse-lattice fermion propagator between non-coinciding points is independent of the chiral disorder fields.

## APPENDIX F: NON-LOCALITY OF THE INTERACTING THEORY AT NON-ZERO LATTICE SPACING

In the free theory, a local *square-root* operator may be constructed at non-zero lattice spacing in the massive case [56]. This is not possible in the interacting case: the fourth-root theory, or the square-root theory for that matter, are non-local for any non-zero fine-lattice spacing  $a_f$ . This paper shows that the magnitude of all the non-local terms, (but not their range!) goes to zero with the fine-lattice spacing.

Here I briefly repeat the argument why, in the interacting fourth-root theory, the range of the non-local terms must be a physical scale [15]. With Eqs. (2.8) and (2.12) in mind let us assume on the contrary that, after  $n + 1$  blocking steps, a local fourth root exists in the sense that

$$\det^{1/4}(D_n) = \exp(-\tfrac{1}{4}\delta S_{eff}) \det(\tilde{D}), \quad (\text{F1})$$

where  $\delta S_{eff}$  is local, and where  $\tilde{D}$  is a local lattice Dirac operator which describes one quark in the continuum limit.<sup>40</sup>

Let us now compare the actual Goldstone-boson spectrum of the ordinary staggered theory (no roots) to that dictated by Eq. (F1). Substituting the fourth power of Eq. (F1) back into Eq. (2.20) and noting that  $\det^4(\tilde{D}) = \det(\tilde{D} \otimes \mathbf{1})$ , the assumed locality of  $\delta S_{eff}$  would imply that the RG-blocked theory is in fact a local four-taste theory with an exact  $U(4)$  taste symmetry. This would imply, in turn, that the fifteen pseudo-Goldstone pions must be exactly degenerate. This conclusion is wrong, however! As explained above (see in particular Sec. IIC and App. D), the RG-blocked theory has the same low-energy spectrum as the original staggered theory. This spectrum constitutes of fifteen *non*-degenerate pseudo-Goldstone pions for any non-zero fine-lattice spacing [9, 34, 57]. Thus, the different lattice symmetries of the staggered theory and of the putative theory defined by the Dirac operator  $\tilde{D} \otimes \mathbf{1}$  rule out a local  $\delta S_{eff}$ . As discussed in Sec. IIE and on, however, the notion of a reweighted theory, namely of a taste-symmetric theory that only *approximates* the staggered theory, can be very useful.

## APPENDIX G: THE PROPAGATOR BOUNDS

In this appendix I collect a few observations on the propagator bounds (3.8) and (5.2). The first thing to notice is that the precise form of these bounds is not important, so long as it is known that the norm of the inverse Dirac operator in question is bounded by some non-zero constant in the limit  $a_f \rightarrow 0$ . That constant will depend on  $m_r(a_c)$ , and may depend on  $a_c$  and  $\Lambda$  as well.

A configuration for which the bound (3.8) is nearly, but, not quite, saturated is an instanton with size  $\rho \sim a_c$ . The bound is not fully saturated because there is no index theorem for staggered fermions. (For related observations, see Refs. [15, 19]; for related numerical work, see Ref. [29].) For zero modes of larger-size instantons, the bound (3.8) may be corrected by factors of  $\log(\rho/a_c)$  due to both wave-function and mass renormalizations

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<sup>40</sup> Both  $\delta S_{eff}$  and  $\tilde{D}$  may in general depend on the original as well as on all the blocked gauge fields. The arguments simplify a bit if no blocking steps are done, as was assumed in Ref. [15].

over the range from  $a_c$  to  $\rho$ . Because  $a_c \leq \rho \leq \Lambda^{-1}$ , and both  $a_c$  and  $\Lambda$  are held fixed, I have neglected such logarithmic corrections.

My remaining comments concern the bound (5.2) in the reweighted theory. (The bound (5.2) pertains to the one-taste reweighted theory derived from the fourth-root staggered theory, but similar comments apply to the four-taste reweighted theory derived from the ordinary staggered theory.) The  $U(1)_\epsilon$  symmetry of the massless staggered theory is disguised as a GWL chiral symmetry by the RG blocking. This chiral symmetry is broken only by the (staggered) mass term, which, in turn, is protected against additive mass renormalization. The same is not true for the reweighted theories: for fixed  $a_f > 0$ , the taste-invariant operator  $D_{inv,n}$  has no chiral symmetry in the limit where the staggered mass goes to zero. Thus, Eq. (5.3) reflects the presence of an additive fermion-mass renormalization in the reweighted theory.

One should, however, be careful with the interpretation of Eq. (5.3). First, as noted above, already in the staggered theory itself the bound (3.8) is not expected to be completely saturated. Second, in this paper I do not consider the reweighted theories as “stand-alone” coarse-lattice theories. Their renormalization is *defined* with references to the underlying fine-lattice cutoff  $a_f$  (see Sec. IV A 2). Thus, the rightmost term in Eq. (5.3) vanishes in the continuum limit  $a_f \rightarrow 0$  with, in particular,  $a_c$  fixed, which implies that *no* fine-tuning of the fermion mass in the reweighted theory is needed.

The conclusion that an additive mass renormalization of a certain size is present cannot, in any case, be drawn based on the magnitude of  $\Delta_n$  alone, as can be seen from the following example. In Ref. [15] another family of reweighted theories was constructed with a taste-invariant Dirac operator  $D_{ov,n}(m) = \tilde{D}_{ov,n}(m) \otimes \mathbf{1}$  such that once again  $D_n(m) - D_{ov,n}(m)$  scales in essentially the same way as does  $\Delta_n(m) = D_n(m) - D_{inv,n}(m)$ . Nevertheless,  $D_{ov,n}(m)$  satisfies a GW relation in the limit where the *staggered* mass  $m$  goes to zero. This implies that, just like the original staggered mass, the fermion mass residing in  $D_{ov,n}(m)$  renormalizes multiplicatively.<sup>41</sup>

In the physical one-flavor theory the only chiral symmetry is anomalous, and instantons modify the quark’s mass. Correspondingly, there is a related tiny correction to the denominator in Eq. (5.2), coming from the integration over instantons with size in the range  $a_f \leq \rho \leq a_c$ . By choosing  $a_c$  small enough, instantons with size  $\rho \leq a_c$  are strongly suppressed, and this correction can be made arbitrarily small relative to  $m_r(a_c)$ . No similar correction exists in a theory with more than one degenerate flavor.<sup>42</sup> For related observations on the fourth-root regularization of one-flavor QCD (prompted by the claims made in Ref. [49]), see Refs. [11, 19].

## APPENDIX H: SCALING AND THE MULTI-GAUGE-FIELD REPRESENTATION

In this appendix I further expand on scaling issues within the multi-gauge-field diagrammatic expansion, discussed in Sec. IV C of the main text. This appendix is organized around a few examples that each illuminate some particular aspect. In App. H 1 I discuss the free

<sup>41</sup> This observation is relevant for Ref. [58] where a comparison of the staggered ensemble to a reweighted overlap ensemble was attempted. See also Ref. [15].

<sup>42</sup> I thank Mike Creutz for a discussion of this point.

theory, and in App. H2 the interacting theory.

### 1. Free theory: small-momentum expansion of $D_n$

Extending the result obtained in Ref. [20] to  $m \neq 0$  and adapting to the present conventions and notation, the free propagator takes the form

$$D_n^{-1} = \mathcal{M}_n - \sum_{\mu} \left( i[\gamma_{\mu} \otimes \mathbf{1}] \mathcal{A}_{\mu}^n + [\gamma_5 \otimes \xi_5 \xi_{\mu}] \mathcal{B}_{\mu}^n \right), \quad (\text{H1})$$

where, in the massless limit,

$$\mathcal{M}_n \Big|_{m=0} = R_n = \sum_{k=0}^n (16)^{k-n} / \alpha_k. \quad (\text{H2})$$

A straightforward calculation gives<sup>43</sup>

$$\mathcal{A}_{\mu}^n = \frac{p_{\mu}}{p^2 + m^2} \left( 1 + O(p^2) \right), \quad (\text{H3a})$$

$$\mathcal{B}_{\mu}^n = 2^{-n-1} a_c \frac{p_{\mu}^2}{p^2 + m^2} \left( 1 + O(p^2) \right). \quad (\text{H3b})$$

Inverting Eq. (H1) we find

$$D_n(p) = m + i[\not{p} \otimes \mathbf{1}] + a_f \sum_{\mu} [\gamma_5 \otimes \xi_5 \xi_{\mu}] p_{\mu}^2 - R_n \left( m + i[\not{p} \otimes \mathbf{1}] \right)^2 + \dots \quad (\text{H4})$$

The ellipsis stand for terms of homogeneity degree three or higher in  $p_{\mu}$  and  $m$ . Observe that, even though  $D_n$  lives on the coarse lattice, the first three terms on the right-hand side of Eq. (H4) are exactly the same as in the usual taste-basis Dirac operator [5, 6]. In particular, the “skewed Wilson term” (that comprises the leading taste-breaking term) has a coefficient that scales with the fine lattice spacing  $a_f = 2^{-n-1} a_c$ . Because  $-\pi/a_c \leq p_{\mu} \leq \pi/a_c$ , the leading taste-breaking term is indeed of order  $a_f/a_c^2$ . Considering the massless limit we see that there is also a term  $R_n p^2$  with the same structure as an ordinary Wilson term. The coefficient  $R_n$  scales with the coarse-lattice spacing because  $\alpha_k^{-1} = O(a_k)$  by assumption. The  $R_n$ -dependent terms reflect the fact that  $D_n$  satisfies a GW relation in the massless limit.

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<sup>43</sup> Equation (H3) follows from Eq. (11) of Ref. [20]. The free propagator  $D_n^{-1}$  is constructed as a sum over terms with fine-lattice momentum  $p + (2\pi/a_c)k^{(n)}$ , where the coarse-lattice momentum  $p$  is fixed, and  $k_{\mu}^{(n)}$  takes integer values such that the  $k_{\mu}^{(n)}$ -summation samples all of the Brillouin zone of the original fine lattice. Thanks to the suppression provided by the blocking kernels, cf. Eq. (11d) therein, any term with  $k_{\mu}^{(n)} \neq 0$  is  $O(p^2)$  at most.

## 2. Aspects of scaling in the interacting theory

The scaling of  $\Delta_n$  was derived in Sec. IV C for any theory whose partition function can be cast in the multi-gauge-field form of Eq. (4.2). This includes as special cases the ordinary and fourth-root staggered theories, with or without reweighting. Here I illustrate some of the “inner working” of the scaling of  $\Delta_n$ . I first discuss two terms that (should) occur in  $\Delta_n$ , and how their functional form depend on the blocking level  $n$ . I then discuss how the two terms scale.

The free-theory result (H4) together with gauge invariance requires the presence in  $\Delta_n$  of a covariant, skewed Wilson term

$$\mathcal{O}_D^{(n)} = a_f z_n^{-2} \sum_{\mu} \bar{\psi}^{(n)} [\gamma_5 \otimes \xi_{\mu} \xi_5] \nabla_{\mu}^{[n]} \psi^{(n)}, \quad (\text{H5})$$

where, using Eq. (2.10b) and the overall normalization of the blocking kernels in Eq. (A5), the wave function renormalization factor is

$$z_n = \prod_{k=1}^n z^{(k)}. \quad (\text{H6})$$

The covariant laplacian  $\nabla_{\mu}^{[n]}$  reduces to  $p_{\mu}^2 + O(p^4)$  in the free-theory limit. The superscript notation  $[n]$  is meant to remind us that, excepting  $\mathcal{V}^{(n)}$ , this covariant laplacian depends on the entire “tower” of gauge fields,  $\nabla_{\mu}^{[n]} = \nabla_{\mu}^{[n]}(\mathcal{U}, \mathcal{V}^{(0)}, \dots, \mathcal{V}^{(n-1)})$ . The explicit (complicated!) form can in principle be computed using Eq. (2.10).

Another example is based on the result of Ref. [32], where it was shown that the taste-basis Dirac operator of the  $k = 0$  step (cf. Eq. (2.10a) and Sec. IV C) contains a term with the generic form

$$\mathcal{O}_F^{(0)} = a_f \bar{\psi}_{\alpha i}^{(0)} \mathcal{F}_{\mu\nu}^{[0]} \psi_{\beta j}^{(0)} c_{\alpha\beta ij\mu\nu}^{(0)}. \quad (\text{H7})$$

The notation  $\mathcal{F}_{\mu\nu}^{[0]} = \mathcal{F}_{\mu\nu}^{[0]}(\mathcal{U})$  is a shorthand for  $(1 - \mathcal{W}_{\mu\nu}(\mathcal{U})) / (ia_f g_0)$ , where  $\mathcal{W}_{\mu\nu}(\mathcal{U})$  is a Wilson-loop operator (without a color trace), and  $g_0$  is the bare coupling;  $\mathcal{F}_{\mu\nu}^{[0]}$  reduces to  $F_{\mu\nu}$  in the classical continuum limit. In Eq. (H7) all indices except color are explicitly shown, and pairs of indices are summed over. The dimensionless tensor  $c_{\alpha\beta ij\mu\nu}^{(0)}$  has  $O(g_0)$  entries, and its explicit form is such that the operator  $\mathcal{O}_F^{(0)}$  violates taste symmetry.

The presence of both  $\mathcal{O}_D^{(0)}$  and  $\mathcal{O}_F^{(0)}$  in the taste-basis Dirac operator is dictated by shift symmetry, which mixes the leading, dimension-four, taste-invariant part of this Dirac operator with taste non-invariant terms of dimension five and higher.<sup>44</sup> The precise form of  $\mathcal{O}_D^{(0)}$  and  $\mathcal{O}_F^{(0)}$  depends on the covariant “blocking” kernel (A2) that has been chosen for the  $k = 0$  step. While Ref. [32] discusses the taste-basis Dirac operator in the case  $\alpha_0 \rightarrow \infty$ , the result (H7) generalizes to  $\alpha_0 < \infty$ .

How the operator  $\mathcal{O}_F^{(0)}$  “evolves” with blocking is less certain than in the case of  $\mathcal{O}_D^{(n)}$ , where we could appeal to gauge invariance (and the free theory) to determine the overall

<sup>44</sup> The taste-symmetric propagator usually used in staggered perturbation theory is related to the taste-basis propagator by a non-local unitary transformation.

normalization. Still, based on the fact that the initial taste-basis operator  $D_0$  is known to contain  $\mathcal{O}_F^{(0)}$ , one would expect that  $D_n$  contains a similar-looking taste-breaking term,

$$\mathcal{O}_F^{(n)} = a_f \bar{\psi}_{\alpha i}^{(n)} \mathcal{F}_{\mu\nu}^{[n]} \psi_{\beta j}^{(n)} c_{\alpha\beta ij\mu\nu}^{(n)}, \quad (\text{H8})$$

where the coefficients  $c_{\alpha\beta ij\mu\nu}^{(n)}$  evolve logarithmically. The operator  $\mathcal{F}_{\mu\nu}^{[n]}$  has similar properties to  $\mathcal{F}_{\mu\nu}^{[0]}$ , except that now it depends on the “tower” of gauge fields  $\mathcal{F}_{\mu\nu}^{[n]} = \mathcal{F}_{\mu\nu}^{[n]}(\mathcal{U}, \mathcal{V}^{(0)}, \dots, \mathcal{V}^{(n-1)})$ .

Let me now consider the contribution of  $\mathcal{O}_F^{(n)}$  to the taste-violating part of blocked observables. The point to make is that any operator of the general form (H8) will be suppressed by the (effective) gauge-field action in the Boltzmann weight (cf. Eqs. (4.2) and (4.4)). The underlying reason that this works is that the discussion below is nothing but a reconstruction, using the terminology of non-perturbative ensemble averages, of the familiar diagrammatic argument why taste-violating processes mediated by a hard-gluon exchange are suppressed by powers of the lattice cutoff [11, 31, 33].

Suppose that a “big part” of  $\mathcal{F}_{\mu\nu}^{[n]}$ , which accounts for the gauge-field dependence of  $\mathcal{O}_F^{(n)}$ , comes from the fine-lattice gauge field, or from a blocked gauge field  $\mathcal{V}^{(k)}$  with  $k \ll n$ . Then one can devise gauge-field configurations for which  $\mathcal{O}_F^{(n)}$  is too large. In fact, there exist configurations for which  $\mathcal{O}_F^{(n)}$  will be  $O(1/a_f)$ . However, all such configuration are rare. A particular example consists of a fine-lattice vector potential  $A_{\mu,x}$  with the shape of a wave packet whose average momentum is  $p \sim 1/a_f$  and whose width is  $\Delta p \sim 1/a_c$ . Such a vector potential is coherent over the coarse-lattice scale; its amplitude,  $\hat{A}$ , can in principle get as large as  $O(1/a_f)$ . Were it to happen, this would give rise to an  $O(1/a_f)$  value of  $\mathcal{O}_F^{(n)}$ . However, large values of  $\hat{A}$  are suppressed. By expanding the fine-lattice gauge-field action to quadratic order one finds that the action of the “wave packet” is  $\sim \hat{A}^2 a_c^4/a_f^2$ . Therefore the average value of  $\hat{A}$  is  $O(a_f/a_c^2)$ . (I have neglected the coupling-constant dependence, together with any other corrections that scale logarithmically with the lattice cutoff.) This, in turn, implies that  $\mathcal{O}_F^{(n)} = O(a_f/a_c^2)$  as well. Obviously, this particular “wave packet” configuration is further suppressed in the ensemble because it has a limited phase space. But a similar conclusion is reached for generic fluctuations of the fine-lattice gauge field, when taking into account that these fluctuations are uncorrelated over the coarse-lattice scale. Individual (local) fluctuation will typically be  $O(1/a_f)$ . The random-walk sum of  $O((a_c/a_f)^4)$  such fluctuations is  $O(a_c^2/a_f^3)$ , if the sum is over a fine-lattice region with roughly the size of a coarse-lattice hypercube. The average value, which is the only thing that a coarse-lattice field will be sensitive to, is  $O(a_f/a_c^2)$ . Thus, again, one finds that  $\mathcal{O}_F^{(n)} = O(a_f/a_c^2)$ . Last, in the case of the local four-taste staggered theory it is clear that, upon integrating the tower of gauge fields, and the fine-lattice gauge field in particular,  $\mathcal{O}_F^{(n)}$  will induce four-fermi (or higher dimensional) terms in the coarse-lattice action  $S_n$ , which are suppressed by  $a_f^2$  (at least).<sup>45</sup>

I now turn to the role of  $\mathcal{O}_D^{(n)}$ . Recall that, according to Sec. IV C,  $\Delta_n$  scales like  $a_f/a_c^2$  (up to logarithms) on the ensemble of any theory defined by Eq. (4.2). However, one cannot

<sup>45</sup> A parallel statement in the context of the fourth-root theory would only be meaningful within a diagrammatic expansion augmented by the replica trick. I stress that the discussion of Sec. V is non-perturbative, and free of this limitation.

deduce that  $\mathcal{O}_D^{(n)}$  scales like  $a_f/a_c^2$  all by itself. We only know that the sum of all contributions to  $\Delta_n$ , coming not only from  $\mathcal{O}_D^{(n)}$  but from many other (higher-dimensional) terms, must scale like  $a_f/a_c^2$ . As I will now explain, this scaling depends not only on the suppression provided by the gauge-field action (as was the case for  $\mathcal{O}_F^{(n)}$ ), but also on the underlying staggered symmetries and, in particular, on shift symmetry. The example below also shows that, in other circumstances, there exist operators that would fit the description of  $\mathcal{O}_D^{(n)}$  as given around Eq. (H5), and yet they would scale as badly as  $1/a_f$ .

In order to illustrate in what way things could be different, let us consider the proposal [54] to couple the taste-basis fermions directly to a gauge field on the lattice with spacing  $a_0 = 2a_f$ . The resulting Dirac operator  $D_0^{DK}$  has no shift symmetry (the superscript “DK” stands for Dirac-Kähler formulation, which was the main thrust of Ref. [54]). The one-loop calculation of Ref. [53] proves that a taste-violating  $O(1/a_f)$  mass term is induced in that theory. Thus,  $\Delta_n^{DK}$  (defined in analogy with Eq. (2.21)) will scale like  $1/a_f$  on the corresponding ensemble, and will diverge in the limit  $a_f \rightarrow 0$ . In more detail, before doing any blocking steps, the  $1/a_f$  divergence originates directly from the skewed Wilson term [53]. By itself, RG blocking obviously cannot “eliminate” any of the divergences of the underlying theory. (Assuming on the contrary that a divergent mass term is present in the fine-lattice propagator, but absent from the blocked propagator, one reaches a contradiction by invoking the pull-back mapping, cf. App. D.) Thus, after  $n$  blocking steps, the  $O(1/a_f)$  scaling is expected to come from an operator  $\mathcal{O}_D^{(n)DK}$  with the same general form as in Eq. (H5). Note that, because the kinetic term is only marginal, the range of the blocked propagator  $1/D_n^{DK}$  rapidly tends to zero with  $n$ . This is, of course, nothing but the decoupling of a fermion with a cutoff-scale mass.

Let us add to the Dirac operator of the DK theory a (taste non-symmetric) mass counterterm:

$$D_{sub}^{DK} = D_0^{DK} + \mathcal{O}_M^{DK}, \quad \mathcal{O}_M^{DK} = m^{DK} \sum_{\mu} [\gamma_5 \otimes \xi_{\mu} \xi_5], \quad (\text{H9})$$

where  $m^{DK} = O(1/a_f)$  too; we moreover fine-tune  $m^{DK}$  so that taste symmetry is restored, and the correct physical quark masses are obtained in the continuum limit. (Any additional, taste-symmetric mass term present in  $D_0^{DK}$  will renormalize multiplicatively [8, 53].) With the subtracted operator  $D_{sub}^{DK}$  at the starting point, the taste-violating effects of the DK theory have become *irrelevant*. The desired  $O(a_f/a_c^2)$  scaling of the taste-breaking  $\Delta_{sub,n}^{DK}$  will now be recovered on the corresponding ensemble.

The example of the DK theory illustrates that there are two distinct issues here: separation of relevant and irrelevant operators; and the scaling of irrelevant operators (in particular, within the current RG framework) once we have actually determined what they are. The separation of relevant from irrelevant terms will in general require subtractions (i.e. additive renormalizations). This is indeed the case in the DK theory. No such subtractions are needed in the staggered theory, thanks to its extended symmetry.

As soon as the appropriate counterterms have been added to the underlying theory, all terms in the RG-blocked lattice action that break any of the symmetries of the continuum theory must have become irrelevant, and will scale accordingly on the corresponding ensemble.<sup>46</sup> This amounts to a standard lore in the case of local theories. In this paper I have

<sup>46</sup> In a strict technical sense, this statement directly applies to internal symmetries. The role of rotation symmetry in the RG-blocked action is more involved, see App. D and App. E.

extended this conclusion to the fourth-root theory (under plausible assumptions).

In summary, what the example of the DK theory shows is that the only conceivable way for the scaling of  $\Delta_n$  to go wrong, is when we overlook some of the necessary counterterms of the underlying theory. Once all the counterterms needed for the desired (universal) continuum limit are introduced, the anticipated scaling of the taste-breaking part of the blocked Dirac operator, as discussed in Sec. IV C, is a generic property of the RG transformation. In the staggered case, however, it turns out that no counterterms are necessary [8], thanks in particular to shift symmetry.

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